Real and Functional Analysis

*Marco Di Francesco**

University of L'Aquila (Italy)

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*marco.difrancesco@univaq.it

Why functional analysis?

Recall a typical problem from linear algebra.

0.1 PROBLEM. Let A be an $n \times n$ matrix with real entries, and let $y \in \mathbb{R}^n$ be a given vector. Find all the vectors $x \in \mathbb{R}^n$ such that

$$Ax = y. \tag{1}$$

The vector x has to be determined. Borrowing geometrical terminology we call the vectors x, y *points*. The proper framework for this problem is that of *vector spaces*, or *linear spaces* ($\mathbb{R}^n$) and *linear mappings*, or *linear operators* (A).

Now note that a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ can be viewed as a *function* $x : \{1, \dots, n\} \to \mathbb{R}$, i.e.

$$\mathbb{R}^n = \{x : \{1, \dots, n\} \to \mathbb{R}\}.$$

So our *points* are actually *functions on a finite set*. *Linear algebra* is the proper setting to deal with them.

> Differently from Linear Algebra, Functional analysis deals with problems (equations) where the sought-after object (unknown) is a function on an *infinite set*.

Let us look at a more specific example.

0.2 PROBLEM. Let $\alpha, y_0 \in \mathbb{R}$ and assume we are given a continuous function $g : [0, +\infty) \to \mathbb{R}$. Find all the differentiable functions $y : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} y'(t) = \alpha y(t) + g(t) & \text{for all } t \geq 0 \\ y(0) = y_0 \end{cases}$$

This is a Cauchy problem for a linear differential equation. Integrating the differential equation on the interval $[0, t]$ for an arbitrary $t > 0$ we get

$$y(t) - \alpha \int_0^t y(s) ds = y_0 + \int_0^t g(s) ds. \tag{2}$$

The unknown is now a *differentiable function* y on the half line $[0, +\infty)$. Can we view the set of differentiable functions on $[0, +\infty)$ as a linear space and regard its elements y as *points in the space*? Can we consider the mapping

$$y = y(t) \quad \mapsto \quad A[y] = A[y](t)\,, \quad A[y](t) = y(t) - \alpha \int_0^t y(s) ds$$

as a linear operator acting on the 'point' y of such a linear space? If so, then the expression (2) could be written as

$$A[y](t) = h(t)\,, \qquad h(t) \doteq y_0 + \int_0^t g(s)ds,$$

and the latter expression is reminiscent of (1). The main different with (1) is that this time the unknown y is a function on an infinite set, namely the half-line $[0, +\infty)$.

> Functional analysis provides the best setting to adapt the notions of *linear space* and *linear operator* to *functions on infinite sets*.

In the example of problem 0.2, the candidate linear space to work with is the space of differentiable functions on the half line $[0, +\infty)$. This is actually a subset of a wider set, the set of *continuous functions* on $[0, +\infty)$. Let us focus on this set for a moment. For simplicity, we replace the half line $[0, +\infty)$ with the closed interval $[0, 1]$.

0.3 FACT. The set of continuous functions on $[0, 1]$ is a *linear space*. The concept of linear space should be well-known to the student. However, we will redefine it later on in this course. For the moment, think of a linear space as a set, the elements of which are called *vectors*. On such set we are allowed to take *sums* between two (or more) vectors and to *multiply* a vector by a real number. In both cases, the operations should produce an *element of the same set* as an outcome. Let us check that two such operations are possible on the set of continuous functions on $[0, 1]$, that we denote by $C([0, 1])$ from now on. We must equip such set with two operations, namely *sum between vectors* and *product of a vector by a real number*. Given $f, g \in C([0, 1])$, we introduce $f + g \in C([0, 1])$ defined by

$$(f + g)(x) = f(x) + g(x). \tag{3}$$

Given $\lambda \in \mathbb{R}$ and $f \in C([0, 1])$, we introduce $\lambda f \in C([0, 1])$ defined by

$$(\lambda f)(x) = \lambda f(x). \tag{4}$$

Let us explain the above two definitions a little more thoroughly. In (3) there are two $+$ (plus) signs. Although they might apparently look alike, there is a subtle difference between them. The plus sign on the right hand side represents an operation involving the two real numbers $f(x)$ and $g(x)$. Here x is a generic real number in $[0, 1]$. The outcome of such operation is the real number $f(x) + g(x)$. So, this is a well-known operation. On the other hand, the plus sign on the left hand side is a new object that we are introducing: it involves two functions f and g, and it allows to build up a new function $f + g$. The formula (3) *defines* such new function, in that is declares how $f + g$

acts on a generic $x \in [0,1]$ and produces the image $(f+g)(x)$. Such image is defined in terms of something totally familiar to all of us, i.e. the sum of two real numbers. A similar situation occurs in (4). Here the well known operation is the multiplication between the two real numbers λ and $f(x)$, whereas the new operation is the multiplication between a real number λ and a function f. A key issue that we haven't checked is that in both (3) and (4) the outcome is an element of the set of continuous functions on $[0,1]$. This is an easy exercise for a student familiar with the concept of continuous function. For the sake of clarity, we shall define the notion of continuous function later on in a more general framework. Finally, we should not forget that every linear space should be equipped with a (unique) *zero* element, i.e. with a unique element 0 such that $f+0=f$ for all f in the linear space. In the case of $C([0,1])$, the zero element is the *constant zero function* $f(t)=0$ for all $t \in [0,1]$.

0.4 FACT. The linear space $C([0,1])$ defined above has *dimension* $+\infty$, or more precisely there exists no integer n such that the dimension of $C([0,1])$ equals n. Let us first recall from linear algebra what the dimension of a linear space is: it is the largest possible integer number N of linearly independent vectors, i.e. the largest possible number of vectors $v_1, \ldots, v_N$ for which $\alpha_1 v_1 + \ldots + \alpha_N v_N = 0$ implies $\alpha_1 = \ldots = \alpha_N = 0$, i.e. the largest possible number of vectors in which none of them can be written as a linear combination of the others. Now, how to prove that the dimension of $C([0,1])$ is *infinite*? The best option is to assume that it is finite and get to a contradiction, which would prove the assertion. So, let us assume that the dimension of our linear space $C([0,1])$ is given by some (possibly very large) integer $N \in \mathbb{N}$. Consider the set of points

$$v_0(t) \doteq 1, \quad v_1(t) \doteq t, \quad v_2(t) \doteq t^2, \quad \ldots, v_N(t) \doteq t^N.$$

We claim that the above $N+1$ points are linearly independent. Assume

$$\sum_{i=0}^{N} \alpha_i t^i = 0 \tag{5}$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_N \in \mathbb{R}$. Note that (5) must hold for all $t \in [0,1]$, so in particular for $t=0$. This implies $\alpha_0 = 0$ upon substituting $t=0$ in (5). Now divide (5) by $t>0$, which gives

$$\sum_{i=1}^{N} \alpha_1 t^{i-1} = 0$$

for all $t>0$. Now, although we cannot substitute $t=0$ above, we can send to the limit $t \to 0$. It is easy to see that we get $\alpha_1 = 0$. Iterating this procedure we prove that $\alpha_2 = \ldots = \alpha_N = 0$. Hence, the $N+1$ monomials $v_0, \ldots, v_N$ are linearly independent. But that contradicts the fact that the dimension of the space was N, because we have found $N+1$ linearly independent vectors. Now, here N was an arbitrary integer. This proves the assertion.

We just "touched the main point about functional analysis with bare hands". Linear spaces of functions (such as the space $C([0,1])$ in the above examples) do not fit the classical background of linear algebra that we learned in our bachelor studies, in which *linear spaces were always assumed to be finite dimensional*. This had a lot of implications, most importantly *linear mappings could be expressed via matrices* with a finite number of entries.

Functional analysis is the extension of linear algebra to *linear spaces with infinite dimension*. Such spaces are typically spaces of functions on infinite sets, often functions on subsets of $\mathbb{R}$ or subsets of the Euclidean space $\mathbb{R}^n$.

Let us now introduce another important issue regarding functional analysis, which is that of *distance measuring*. It is well known that one can measure *distances* in a finite dimensional linear space. In the most intuitive case, namely the Euclidean space $\mathbb{R}^n$, the distance between two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by the canonical formula

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

Other non-canonical ways could be

$$d_1(x,y) = |x_1 - y_1| + \ldots |x_n - y_n|,$$

or

$$d_\infty(x,y) = \max_{i=1,\ldots,n} |x_i - y_i|.$$

We could measure distances between points using all of those distances, and of course they might (except in the case $n = 1$) give rise to different concepts of distance. A concept of distance easily defines a concept of *closeness*: very intuitively, two points are close if their distance is small. A rigorous way to introduce a notion of closeness is via the concept of limit, which is the basic concept of mathematical analysis. Again, this concept should be well known to the student. It will be re-defined for convenience below. In rough words, a sequence (i.e. an infinite set $v_1, v_2, v_3, \ldots, v_n, \ldots$ indexed by integer numbers) of points in a linear space *converges* to a point v if the distance between v_n and v tends to zero as $n \to +\infty$. Now, clearly *two separate notions of distances may give rise to two separate notions of closeness, or convergence*. However, here is an important fact!

0.5 FACT. Given a sequence of points $\{x_k\}_{k=1}^{+\infty} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$. Then $d_1(x_k, x) \to 0$ as $k \to +\infty$ if and only if the same holds with d_2 or d_∞ replacing d_1. In other words, the notion of *convergence of a sequence* in $\mathbb{R}^n$ is *not affected* by the choice of the 'distance' (d_1, d, or d_∞). Another way to see it: 'being close' in the d_1 sense is equivalent to being close in the d sense of in the d_∞ sense. Proving this fact is not too difficult, but we shall omit the details at this stage. The main idea is the following: think of $x = (x_1, \ldots, x_n)$ and $x_k = ((x_k)_i, \ldots, (x_k)_n)$ as vectors in $\mathbb{R}^n$; having x_k and x very close in any of the three distances under consideration is equivalent to having the i-th components $(x_k)_i$ and x_i very close (as real numbers!) for all $i = 1, \ldots, n$. The bottom line, here, is the following: *in finite dimensional spaces, the concept of closeness is independent of the distance we are using.*

> In finite dimensional spaces (more precisely, on finite dimensional *normed* spaces, this concept will be introduced later on), different ways of measuring distances generate the same notion of *closeness*. If two points are close to each other in a finite dimensional space, this does not depend on the type of distance we are using. *This is, in general, not true in infinite dimensional spaces.* Therefore, having to deal with infinite dimensional spaces, *functional analysis explores many possible ways to measure distances between points (functions).*

It is instructive to produce an example of two separate notions of closeness already at this stage in infinite dimensions.

0.6 EXAMPLE. Consider the space of continuous functions on $[0, 1]$ denoted by $C([0, 1])$. For each $n \in \mathbb{N}$ we define the function

$$f_n(x) = x^n, \qquad x \in [0, 1].$$

We can actually see f_n as a *sequence of functions*, i.e. a family of functions indexed by a positive integer n. We may ask ourselves whether or not this sequence converge to a limit, i.e. to a function $f \in C([0, 1])$. As explained above, this notion has a lot to do with the concept of distance we are using. The goal of this example is to show that f_n has a limit with respect to some distance, and has not with respect to some other one. For a given $g \in C([0, 1])$ let us define

$$\|f\|_\infty := \max_{x \in [0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)| dx.$$

Since $x^n \to 0$ for all $x \in [0, 1)$, the most reasonable candidate limit for this sequence (no matter what distance we are using) is

$$f \equiv 0.$$

Let us compute

$$\|f_n - 0\|_\infty = \max_{x \in [0,1]} |f_n(x)| = \max_{x \in [0,1]} |x^n| = 1.$$

Therefore, f_n does not converge to zero in the ∞ distance. On the other hand,

$$\|f_n - 0\|_1 = \int_0^1 x^n dx = \frac{1}{n+1} \to 0$$

as $n \to +\infty$.

The student may, at this stage, wonder why distance measuring between functions is so important. An easy answer comes, for instance, from numerical methods for differential equations, a matter of massive impact in the applications. While working on a numerical scheme for a differential equation, one needs to know whether or not the method is a *good approximation* of the solution to the equation. How do we measure the approximation? We need to be able to establish whether or not the solution to our numerical scheme is *close* to the actual solution to the equation. Therefore, first of all we should define what we mean by 'close'. We shall get back to this point later on in this section.

Before Fact 0.5, the only subject we invoked as a background was linear algebra. All of a sudden, in Fact 0.5 we started dealing with sequences, limits, etc, things we were used to in *calculus and analysis*. Functional analysis is indeed a lot about analysis, not just linear algebra and linear mappings. In fact,

> Functional analysis is a subject in which *analysis and linear algebra merge together*.

So far we provided a partial answer to the question 'Why functional analysis?'. We tried somehow to justify functional analysis as a natural continuation of a pedagogical path based on linear algebra and analysis. But is all that of any use? Problem 0.2 is a good start, as it involves differential equations, an undoubtedly useful tool in science and engineering. However, in many practical situation we deal with *optimisation problems*, that is, given a variable quantity, we want to compute its maximum or minimum value given a set of *constraint*. Optimisation is probably the most important subject in industrial applied mathematics.

0.7 fact. In finite dimensions, a continuous function $f : K \to \mathbb{R}$ defined on a bounded and closed set $K \subset \mathbb{R}^n$ has a maximum and a minimum. This is a famous theorem in analysis due to Weierstrass. It is quite useful while looking for the solution to an optimisation problem depending on finitely many variable, since it guarantees under quite general assumptions that - no matter how good we are in finding a solution to the problem - there actually *is* a solution. The key issue behind Weierstrass theorem is the fact that a bounded and closed set in finite dimension is always *compact*, i.e. every sequence in K has a convergent subsequence.

0.8 EXAMPLE. In classical mechanics, the trajectory of a material body subject to conservative forces can be found by minimising the quantity

$$L = \int_0^T \Big[m|\dot{x}(t)|^2 + U(x(t)) \Big] \, dt,$$

in which the first term is the *kinetic* energy and second one the *potential* energy. The unknown of the problem is a curve $[0, T] \ni t \mapsto x(t) \in \mathbb{R}^3$ standing for the optimal trajectory, i. e. the unknown is a vector in which each component is a differentiable function. Hence, the *minimiser* of the quantity L above lives in an infinite dimensional space. Weierstrass Theorem no longer holds in infinite dimensions, and ensuring that a quantity of the form of L attains the minimum on some curve $t \mapsto x(t)$ is far from being trivial. This is due to the fact that in infinite dimension *it is much more difficult to prove that a given set is compact*.

One of the most important concepts developed in a functional analysis course is that of *compactness*.

As briefly mentioned above, mathematical modelling in applied sciences often relies on *numerical calculus*, in which (for example) the solution to a differential equation is approximated via some finite elements method.

0.9 EXAMPLE. A differential equation

$$\dot{y}(t) = f(y(t), t), \qquad t \in [0, T],$$

can be approximated via finite difference methods e.g. by the set of equations

$$y_{i+1} = y_i + (\Delta t) f(y_{i+1}, t), \tag{6}$$

with $i \in \{0, \dots, n-1\}$ and $(\Delta t)n = T$. The unknown of the system of equations in (6) is the $(n+1)$-dimensional vector $(y_0, y_1, \dots, y_n)$, whereas the unknown of the differential equation above is a function on $[0, T]$, which lives in an infinite dimensional space. In order to make sure that the numerical scheme (6) works, one has to prove that a suitable interpolation (piecewise constant, or piecewise linear) of the solution $(y_0, y_1, \dots, y_n)$ to (6) gets closer and closer to the solution $y(t)$ of the above differential equation as Δt goes to zero. Once again, it's a matter of measuring a distance between functions. What is the correct distance to use? How can one prove that our numerical scheme *converges*? Functional analysis provides tools to answer to these questions.

The previous example provides another interpretation of the expression *infinite dimensional*. The vector $(y_0, y_1, \dots, y_n)$ in the example is $n+1$ dimensional. The dimension depends on the size of the time-step Δt through the formula $(\Delta t)n = T$. The smaller Δt, the larger n, Such a variability of the dimension is important in order to consider smaller and smaller time-steps while approaching the solution to our numerical problem. *A vector with finite entries but with no constraint on the number of entries can be considered as well as an infinite dimensional vector.* More precisely, we can think of y as a vector with infinite entries $(y_0, y_1, \ldots, y_{N-1}, y_N, 0, 0, \ldots, 0, \ldots)$.

Why has functional analysis become so fashionable? Despite the above examples, it is often the case that functional analysis leads to a result in cases in which other traditional techniques do as well. The typical example is optimisation. A very abstract functional analytical framework may provide a non-constructive existence of a minimiser for a given optimisation problem, but one may directly recover a set of Euler-Lagrange equations as optimality conditions, thus having to solve just a set of differential equations.

However, the strength and appeal of functional analysis is that it is a convenient way of examining the mathematical behavior of various structures. More precisely, functional analysis *clarifies*, *rigorises*, and *unifies* the underlying concepts.

It clarifies because - as already said - functional analysis is a generalisation and combination of linear algebra, analysis, and geometry (yes, there is a bit of geometry too when you measure distances: orthogonal projections, hyper-planes, etc.), expressed in a simple mathematical notation which allows these three aspects of the problem to be easily seen. It rigorises, because it has the back up of a vast mathematical machinery which subsumes many of the classical results on differential equations, numerical methods, calculus of variations, and applied mathematical techniques. It unifies, because often the simple notation does away with many of the complicating details leaving the essential standing out clearly, so that problems from many different fields have the same functional analytical symbolism.

This course has several prerequisites. We try to list them here: set theory, relations and functions, partially ordered sets, the set of real numbers and its properties, supremum and infimum, topology of the real line, matrices, vectors, linear spaces, linear independence, linear systems and their resolution, Euclidean geometry, diagonalisation of matrices, eigenvectors, eigenvalues, topology of Euclidean spaces, real functions of one and more variables, real sequences, limits, derivatives, partial derivatives, real sequences and their limits, lim sup and lim inf, infinite sums and their convergence, basics of ordinary differential equations, Riemann's integration theory.

The present lecture notes are adapted from many references which include [3] as the main reference plus some hand-written notes by the lecturer.

1 Metrics, norms, and topologies

We are all familiar with the geometrical properties of ordinary, three dimensional Euclidean spaces. A persistent theme in mathematics is the grouping of various kinds of objects into abstract spaces. This grouping enables us to extend our intuition of the relationship between points in Euclidean space to the relationship between more general kinds of objects, leading to a clearer and deeper understanding of those objects.

The simplest setting for the study of many problems in analysis is that of a metric space. A metric space is a set of points with a suitable notion of the distance between points. We can use the distance metric, or distance function, to define the fundamental concepts of analysis, such as convergence, continuity, and compactness.

In general, a metric space does not have any kind of algebraic structure defined on it. In many applications, however, the metric space is a linear space, with a metric derived from a *nor*m that gives the 'length' of a vector. Such spaces are called *normed linear spaces*. For example, the n-dimensional Euclidean space is a normed linear space (after the choice of an arbitrary point as the origin). A central topic of this course is to study infinite-dimensional normed linear spaces, including function spaces in which a single point represents a function. As we will see, the geometrical intuition derived from finite-dimensional Euclidean spaces remains essential, although completely new features arise in the case of infinite-dimensional spaces.

In this section we define and study metric spaces and normed linear spaces. Along the way, we review a number of definitions and results from real analysis.

1.1 *Metrics and norms*

Let X be an arbitrary nonempty set.

1.1 definition. A *metric*, or *distance* (or distance function), on X is a function

$$d : X \times X \to \mathbb{R}$$

with the following properties

(a) $d(x,y) \geq 0$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$;

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(c) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

A *metric space* is a pair (X, d) where X is a set and d is a metric on X. The elements of X are called *points*.

When the metric d is understood from the context, we denote a metric space simply by the set X. In words, the definition states that:

(a) distances are nonnegative, and the only point at zero distance from x is x itself;

(b) the distance is a symmetric function;

(c) distances satisfy the *triangle inequality*.

For points in the Euclidean space, the triangle inequality states that the length of one side of a triangle is less than the sum of the lengths of the other two sides.

1.2 example. The set of real numbers $\mathbb{R}$ with the distance function $d(x,y) = |x-y|$ is a metric space. The set of complex numbers $\mathbb{C}$ with the distance function $d(z,w) = |z-w|$ is also a metric space.

1.3 example. Let X be the set of people of the same generation with a common ancestor, for example, al the grandchildren of a grandmother. We define the distance $d(x,y)$ between two individuals $x, y \in X$ as the number of generations one has to go back in order to find the first common ancestor. For example, the distance between two sisters is one. It is easy to check that d is a metric.

1.4 example (Metric subspaces). Suppose (X,d) is any metric space and Y is a subset of X. We define the distance between points of Y by restricting the metric d to Y.[1] The resulting metric space $(Y, d|_Y)$, or (Y,d) for short, is called a *metric subspace* of (X,d). For example, $(\mathbb{R}, |\cdot|)$ is a metric subspace of $(\mathbb{C}, |\cdot|)$, and the space of rational numbers $(\mathbb{Q}, |\cdot|)$ is a metric subspace of $(\mathbb{R}, |\cdot|)$.

1.5 example (Cartesian products). If X and Y are sets, then the *Cartesian product* $X \times Y$ is the set of ordered pairs (x,y) with $x \in X$ and $y \in Y$. If d_X and d_Y are metrics on X and Y respectively, then we may define a metric $d_{X\times Y}$ on $X \times Y$ by

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

1.6 exercise. Let (X,d) be a metric space. Prove that, for all $x,y,z \in X$, one has

$$|d(x,y) - d(x,z)| \leq d(y,z)$$

(Hint: use the triangle inequality).

As mentioned above, metric spaces do not need an underlying algebraic structure to be well defined, see for instance the Example 1.3. When talking about 'algebraic structure', we essentially mean a structure of linear space. In the next definition, we shall refer to $\mathbb{R}$ and $\mathbb{C}$ as 'scalar fields', i.e. sets equipped with a sum operation $+$ and a product operation $\cdot$ with certain

[1] The restriction of a function $f : A \to B$ to a subset $C \subset A$ is the function $f|_C \to B$ defined by $f_C(x) = f(x)$ for all $x \in C \subset A$. In this case, the distance function is restricted to $Y \times Y \subset X \times X$.

elementary properties satisfied[2] such as associativity, commutativity, existence of additive and multiplicative identity elements (zero and one respectively), existence of additive inverses and multiplicative inverses, and distributivity of multiplication over addition. These properties are considered as elementary. When referring to $\mathbb{R}$ or $\mathbb{C}$ as a scalar field, we shall often refer to their elements as *scalar*.

1.7 definition. A *linear space* (or vector space) X over the scalar field $\mathbb{R}$ (or $\mathbb{C}$) is a set, the elements of which are called *vectors*, on which the following two operations are defined

- Sum between vectors: $X \times X \ni (x, y) \mapsto x + y \in X$,[3]
- Scalar multiplication: $\mathbb{R} \times X$ or $(\mathbb{C} \times X) \ni (\lambda, x) \mapsto \lambda x \in X$,

with the following properties:

1. For all $x, y, z \in X$,
   - $x + y = y + x$
   - $x + (y + z) = (x + y) + z$,
2. there exists $0 \in X$ such that $0 + x = x$ for all $x \in X$,
3. for all $x \in X$ there is a unique $-x \in X$ such that $x + (-x) = 0$,
4. for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$),
   - $1\,x = x$
   - $(\lambda + \mu)x = \lambda x + \mu x$,
   - $\lambda(\mu x) = (\lambda\mu)x$,
   - $\lambda(x + y) = \lambda x + \lambda y$.

A norm on an linear space is a function that provides a "length" to a vector.

1.8 definition. A *norm* on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

(a) $\|x\| \geq 0$ for all $x \in X$ (nonnegativity);

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) (homogeneity);

(c) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ (triangle inequality);

(d) $\|x\| = 0$ implies that $x = 0$ (strict positivity).

A *normed linear space* $(X, \|\cdot\|)$ is a linear space X equipped with a norm $\|\cdot\|$.

[2] `https://en.wikipedia.org/wiki/Field_(mathematics)`

[3] By abuse of notation, the same symbol $+$ is used to denote both the sum in the scalar field (sum of two real or complex *numbers*) and the sum in the linear space (sum of two vectors).

Depending on whether X in the above definition is a linear space on the scalar field $\mathbb{R}$ or $\mathbb{C}$, we shall call X a *real* normed linear space or *complex* normed linear space respectively.

1.9 exercise. Some textbooks state the above property (d) as follows:

(d') $\|x\| = 0$ if and only if $x = 0$.

Clearly, (d') implies (d). In fact, (d) and (d') are equivalent once the previous properties (a)-(b)-(c) are assumed. Why?

Clearly, metric spaces and normed spaces do have something in common. The first thing that catches our attention is that a normed space structure needs a linear space structure underneath, whereas a metric space structure can, in principle, be defined on an set which is not a linear space. What if we restrict our survey to linear spaces? Actually, we can prove, in some sense, that a normed space is also a metric space.

1.10 exercise (Normed spaces are metric spaces). Prove that a normed linear space $(X, \|\cdot\|)$ is a metric space with the metric

$$d(x, y) = \|x - y\|. \tag{7}$$

The distance d in the Exercise 1.10 is called *induced distance*.

For future use, we recall the concept of *convex subset*. A subset C of a linear space X is said to be *convex* if

$$tx + (1 - t)y \in C$$

for all $x, y \in C$ and for all $t \in [0, 1]$, meaning that the line segment joining two vectors in C lies entirely in C.

1.11 exercise. Prove that the closed unit ball

$$\{x \in X : \|x\| \leq 1\}$$

in a linear normed space X is a convex set.

1.12 example. The set of real numbers $\mathbb{R}$ with the absolute value norm $\|x\| = |x|$ is a one-dimensional real normed linear space. More generally, $\mathbb{R}^n$, where $n = 1, 2, 3, \ldots$, is an n-dimensional linear space. We define the *Euclidean norm* of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2},$$

and call $\mathbb{R}^n$ equipped with the Euclidean norm *n-dimensional Euclidean space*. As seen in the introductory Section, we can also define other norms on $\mathbb{R}^n$. For example, the 1-*norm* is given by

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|.$$

The maximum norm, or ∞-norm, is given by

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}.$$

The student is invited to prove that the three above (Euclidean norm, 1-norm, and ∞-norm) are *actual* norms according to our Definition 1.8.

1.13 exercise. In the case $n = 2$, draw the sets

- $\{x \in \mathbb{R}^2 : \|x\| \le 1\}$,
- $\{x \in \mathbb{R}^2 : \|x\|_1 \le 1\}$,
- $\{x \in \mathbb{R}^2 : \|x\|_\infty \le 1\}$,

on the Cartesian coordinate system.

1.14 example. A *linear subspace* of a linear space, or simply a *subspace* when it is clear we are talking about linear spaces, is a subset that is itself a linear space. More precisely, A subset M of a linear space X is a subspace if and only if $\lambda x + \mu y \in M$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) and all $x, y \in M$. A subspace of a normed linear space is a normed linear space with norm given by the restriction of the norm on X to M. Note that every subset of a normed linear space X can be therefore seen as a metric sub-space of X (in the induced distance), but not all subsets of X are linear subspaces of X. For example, a bounded line segment of the 2-dimensional Euclidean plane is a metric subspace of $\mathbb{R}^2$, but not a linear subspace.

We will see later on (but we already mention that in section) that all norms on a finite-dimensional linear space lead to the same notion of convergence, so often it is not important which norm we use. Different norms on an infinite-dimensional linear space, such as a function space, may lead to completely different notions of convergence, so the specification of a norm is crucial in this case. We will always regard a normed linear space as a metric space with the metric defined in (7), unless we explicitly state otherwise. Nevertheless, this equation is not the only way to define a metric on a normed linear space, see the Exercises at the end of this Section.

Let us get back to the question raised before Example 1.10. What is the relation between metric spaces and normed linear spaces? So far we have proven that essentially every normed linear space is a metric space. The opposite question raises naturally: suppose X is a linear space which is also a metric space, with metric d; *is the distance d induced by any kind of norm*? If so, every linear space which is also a metric space would be a normed space. The answer is provided in the next example.

1.15 example. On a real vector space X, consider the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

We claim that d cannot induce a norm. By contradiction, assume there exists a norm $\|\cdot\|$ on X such that $\|x-y\| = d(x,y)$. Now, let $x \in X \setminus \{0\}$ (0 being the identity element in X), let $\lambda \in \mathbb{R} \setminus \{0\}$. Then, the homogeneity property of norms implies $1 = d(\lambda x, 0) = \|\lambda x\| = |\lambda|\|x\| = |\lambda| d(x,0) = |\lambda|$, and this is a contradiction if $|\lambda| \neq 1$.

1.2 *Convergence*

The goal of this subsection is to introduce the concept of *convergent sequence* in a metric space.

A sequence (x_n) in a metric space (X,d) is a map $\mathbb{N} \ni n \mapsto x_n \in X$ which associates a point $x_n \in X$ with each natural number $n \in \mathbb{N}$.

1.16 definition. A sequence (x_n) in a metric space (X,d) *converges* to $x \in X$ if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. The point x is called *limit* of the sequence. The sequence is *Cauchy* if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

We use the notations

$$\lim_{n \to +\infty} x_n = x\,, \quad x_n \to x$$

to denote that x_n converges to x.

1.17 remark. The limit of a convergent sequence in a metric space is unique. To see this, assume $x_n \to x$ and $x_n \to y$ as $n \to +\infty$. Assume $x \neq y$. Then, point (a) in the Definition 1.1 implies $d(x,y) > 0$. Let $d(x,y) = \delta > 0$. Since $x_n \to x$, there is an $N \in \mathbb{N}$ such that $d(x, x_n) < \delta/3$ for all $n \geq N$. Hence, due to the triangle inequality, for $n \geq N$ one has

$$\delta = d(x,y) \leq d(x,x_n) + d(x_n,y) < \frac{\delta}{3} + d(x_n,y),$$

which implies $d(x_n, y) > \frac{2\delta}{3}$ for all $n \geq N$, and that contradicts $x_n \to y$.

In the introductory section we emphasized the fact that two separate concepts of distance give rise to two separate notions of convergence. We touched in particular the issue of convergence in a finite dimensional normed space, and outlined that no matter what norm we use in a finite dimensional space, this won't affect the set of convergent sequences. In a generic metric space (not necessarily normed) the fact that the notion of convergence depends quite heavily on the distance is easily seen even in very simple examples.

1.18 example. Let $X = \mathbb{R}$ and consider the following two distances on X:

$$d_1(x,y) = |x-y|$$

$$d_2(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

There are sequences which converge in the metric space (X, d_1) but not in (X, d_2), see also the Exercises. As example, take $x_n = 1/n$. Clearly $x_n \to 0$ in d_1, but this is not true in the distance d_2. Indeed, let $\epsilon = 1/2$. To have convergence we would need to find a $N \in \mathbb{N}$ such that $d_2(1/n, 0) < 1/2$ for all $n \geq \mathbb{N}$. But the only possibility to have $d_2(1/n, 0) < 1/2$ is that $1/n = 0$, which never happens even if n is very large.

1.19 exercise. Prove that every convergent sequence in a metric space is a Cauchy sequence.

The reverse property, namely that every Cauchy sequence converges, singles out a particularly useful class of metric spaces, called *complete* metric spaces.

1.20 definition. A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X. A subset Y of X is complete if the metric subspace $(Y, d|_Y)$ is complete. A normed linear space that is complete with respect to the induced metric is called a *Banach space*.

Recall the set of rational numbers $\mathbb{Q}$, which can be seen as a linear subspace of $\mathbb{R}$, so it is by itself a linear space. Hence, by seeing $\mathbb{R}$ as a normed linear space equipped with the usual absolute value norm, $\mathbb{Q}$ can be seen as a normed linear space. As such, $\mathbb{Q}$ is not complete, since a sequence of rational numbers which converges in $\mathbb{R}$ to an irrational number (such as $\sqrt{2}$ or π) is a Cauchy sequence in $\mathbb{Q}$, but does not have a limit in $\mathbb{Q}$, see the Exercises at the end of this Chapter for a specific example.

1.21 exercise. Prove that the normed linear space $\mathbb{R}^n$ is a Banach space when equipped with the norms $\|\cdot\|$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$ considered in the Example 1.12.

Infinite series (or infinite sums) do not make sense in a general metric space, because we cannot add points together in a general metric space. We can, however, consider series in a normed linear space X. Just as for real or complex numbers, if (x_n) is a sequence in X, then the series $\sum_{n=1}^{+\infty} x_n$ *converges* to $s \in X$ if the sequence of partial sums (s_n), $s_n = \sum_{k=1}^{n} x_k$, converges to s.

The concepts of lim sup and lim inf of a sequence of real numbers are required as prerequisites of this course. We invite the students to review them in a proper basic real analysis textbook. The next property will be useful in the sequel.

1.22 example. If $\{x_{n,\alpha} \in \mathbb{R} : n \in \mathbb{N}, \alpha \in \mathcal{A}\}$ is a set of real numbers indexed by the natural numbers $\mathbb{N}$ and an arbitrary set $\mathcal{A}$, then

$$\sup_{\alpha \in \mathcal{A}} \left[\liminf_{n \to +\infty} x_{n,\alpha} \right] \leq \liminf_{n \to +\infty} \left[\sup_{\alpha \in \mathcal{A}} x_{n,\alpha} \right].$$

Having a concept of distance in our hands, we can introduce the concept of *bounded set* even though a metric space is not necessarily equipped with the structure of totally ordered set. Suppose that A is a nonempty subset of a metric space (X, d). We define the *diameter* of A as

$$\operatorname{diam} A = \sup\{d(x,y) \ : \ x, y \in A\}.$$

A subset A of X is *bounded* if $\operatorname{diam} A$ is finite. It follows that A is bounded if and only if there is an $M \in \mathbb{R}$ and an $x_0 \in X$ such that $d(x_0, x) \leq M$ for all $x \in A$ (see the Exercises at the end of this Chapter for the proof). The *distance* $d(x, A)$ of a point x from the set A is defined by

$$d(x, A) = \inf\{d(x,y) \ : \ y \in A\}.$$

The statement $d(x, A) = 0$ does not imply necessarily that $x \in A$.

Given two metric spaces (X, d_X) and (Y, d_Y), we say that a function $f : X \to Y$ is *bounded* if its range $f(X)$ is bounded. For example, a real-valued function $f : X \to \mathbb{R}$ is bounded if there is a finite number M such that $|f(x)| \leq M$ for all $x \in X$. We say that $f : X \to \mathbb{R}$ is bounded from above if there is an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in X$, and bounded from below if there is an $M \in \mathbb{R}$ such that $f(x) \geq M$ for all $x \in X$.

1.3 *Continuity*

Everyone is familiar with the concept of continuous function $f : \mathbb{R} \to \mathbb{R}$. The definition of continuity for functions between metric spaces is an obvious generalisation of that. Let (X, d_X) and (Y, d_Y) be two metric spaces.

1.23 DEFINITION. A function $f : X \to Y$ is *continuous* at $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$. The function f is *continuous on* X if it is continuous at every point $x \in X$.

If f is not continuous at x, then we say that f is *discontinuous* at x. There are continuous functions on any metric space. For example, every constant function is continuous.

1.24 EXAMPLE (Distance function). Let $a \in X$, and define $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous on X. Indeed, let $x_0 \in X$ and $\epsilon > 0$. As a consequence of the triangle inequality (see the exercise 1.6), we have

$$|f(x) - f(x_0)| = |d(x,a) - d(x_0, a)| \leq d(x, x_0).$$

Therefore, choosing $\delta = \epsilon$ gives $|f(x) - f(x_0)| < \epsilon$ provided $d(x, x_0) < \delta$.

We can also define continuity in terms of limits. If $f : X \to Y$, we say that $f(x) \to y_0$ as $x \to x_0$, or

$$\lim_{x \to x_0} f(x) = y_0,$$

if for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < d_X(x, x_0) < \delta$ implies that $d_Y(f(x), y_0) < \epsilon$. Similarly to the concept of limit for real functions studied in first year's calculus, the above definition does not prescribe any requirement on the value of f on the point x_0. In fact, such a concept can be extended to a point x_0 which is the limit of a sequence on which the function f is well defined. More precisely, let $f : D \to Y$, with $D \subset X$. Let $x_0 \in X$ such that x_0 is the limit of a sequence $(y_n) \subset D$. We say that f has limit y_0 at the point x_0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < d_X(x, x_0) < \delta$ and $x \in D$ implies that $d_Y(f(x), y_0) < \epsilon$. A function $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

meaning that the limit of f as $x \to x_0$ exists and is equal to $f(x_0)$.

If $f : X \to Y$ and E is a subset of X, then we say that *f is continuous on E* if it is continuous at every point $x \in E$. This property is, in general, not equivalent to the continuity of the restriction f_E of f on E, as shown in the next example.

1.25 example. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

The function f is discontinuous at every point of $\mathbb{R}$, but $f|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ is the constant function $f|_{\mathbb{Q}}(x) = 1$, so $f|_{\mathbb{Q}}$ is continuous on $\mathbb{Q}$.

A subtle, but important, strengthening of continuity is *uniform continuity*.

1.26 definition. A function $f : X \to Y$ is *uniformly continuous on X* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

The crucial difference between definition 1.23 and definition 1.26 is that the value of δ does not depend on the point $x \in X$ in the latter, so that $f(y)$ gets closer to $f(x)$ at a uniform rate as y gets closed to x. For example, $r : (0, 1) \to \mathbb{R}$ defined by $r(x) = 1/x$ is continuous on $(0, 1)$ but not uniformly. In the following, we will denote all metrics by d when it is clear from the context which metric is meant.

There is a useful equivalent way to characterise continuous functions on metric spaces in terms of sequences.

1.27 definition. A function $f : X \to Y$ is *sequentially continuous at $x \in X$* if for every sequence $(x_n)_n$ in X that converges to x, the sequence $(f(x_n))_n$ in Y converges to $f(x) \in Y$.

1.28 proposition. *Let X, Y be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if and only if it is sequentially continuous at x.*

Proof. First, we show that if f is continuous, then it is sequentially continuous. Suppose that f is continuous at x, and let $x_n \to x$. Let $\epsilon > 0$ be given.

By the continuity of f, we can choose $\delta > 0$ such that $d(x_n, x) < \delta$ implies $d(f(x_n), f(x)) < \epsilon$. By the convergence or $(x_n)_n$, we can choose N so that $n \geq N$ implies $d(x, x_n) < \delta$. Therefore, $n \geq N$ implies $d(f(x_n), f(x)) < \epsilon$, and this means that $f(x_n) \to f(x_n)$.

To prove the converse, we prove that if f is discontinuous, then it is not sequentially continuous. If f is discontinuous at x, then there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in X$ with $d(x_n, x) < 1/n$ and $d(f(x_n), f(x)) \geq \epsilon$. The sequence (x_n) constructed converges to x but $(f(x_n))$ does not converge to $f(x)$. Hence, f is not sequentially continuous. $\square$

Similarly to what we showed for convergence of sequences, the notion of continuity for a function pretty much depends on the distance one is considering on the metric space.

1.29 example. Let $X = \mathbb{R}$, and let d_1 and d_2 be as in the example 1.18. Consider the function $f : (X, d_1) \to (X, d_1)$ given by $f(x) = x$. This function is continuous, as we all know (a straight line on the real numbers equipped with the classical Euclidean distance). Let us now consider the same function between (X, d_1) to (X, d_2). In order to have f continuous, every converging sequence in (X, d_1) should be mapped via f to converging sequence in (X, d_2). But the only converging sequences with respect to the d_2 distance are those which are eventually constant. So, take the sequence $x_n = 1/n$ converging to 0 in d_1. Its image is $f(x_n) = x_n$, which does not converge to 0 in d_2. So, $f : (X, d_1) \to (X, d_2)$ is not continuous.

There are two kinds of 'half-continuous' real-valued functions, defined as follows.

1.30 definition. A function $f : X \to \mathbb{R}$ is *upper semicontinuous* on X if for all $x \in X$ and every sequence $x_n \to x$, we have

$$\limsup_{n \to +\infty} f(x_n) \leq f(x).$$

A function $f : X \to \mathbb{R}$ is *lower semicontinuous* on X if for all $x \in X$ and every sequence $x_n \to x$, we have

$$\liminf_{n \to +\infty} f(x_n) \geq f(x).$$

1.31 example. Let $X = \mathbb{R}$ equipped with the usual distance $d(x, y) = |x - y|$. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Prove that f is upper semi-continuous at $x = 0$ but not lower semi-continuous at $x = 0$.

1.32 exercise. Prove that a function $f : X \to \mathbb{R}$ is continuous if and only if it is upper and lower semicontinuous.

1.4 *Topological spaces*

The notion of topological space is defined by means of rather simple and abstract axioms. It is very useful as an 'umbrella' concept which allows to use the geometric language and the geometric way of thinking in a broad variety of vastly different situations, which include metric spaces as a special case.

1.33 definition. A *topological space* is a pair (X, τ), where X is a set and a $\tau \subset \mathcal{P}(X)$ is a family of subsets of X called the *topology of* X, whose elements are called *open sets*, such that

(i) $\emptyset, X \in \tau$ (the empty set and the whole set are open sets).

(ii) If $\{O_\alpha\}_{\alpha\in A} \subset \tau$ is an arbitrary family of open sets, then $\bigcup_{\alpha\in A} O_\alpha \in \tau$ (the union of an arbitrary family of open sets is open.

(iii) If $\{O_j\}_{j=1}^N \subset \tau$, then $O_1 \cap \ldots \cap O_N \in \tau$ (the intersection of finite number of open sets is open).

If $x \in X$, then an open set containing x is called an (open) neighborhood of x. The complements of open sets are called *closed sets*.

We will often omit the topology τ, and refer to X as a topological space assuming that the topology has been described.

1.34 example. Let $X = \mathbb{R}$ and let us define as open sets $O \subset \mathbb{R}$ all subsets of $\mathbb{R}$ with the property that, for all $x \in O$, there exists $\varepsilon > 0$ such that the interval $(x-\varepsilon, x+\varepsilon) \subset O$. Then, the above family of open sets defines a topology on $\mathbb{R}$ (exercise!).

1.35 example. If in the set of real numbers $\mathbb{R}$ we declare open (besides the empty set and $\mathbb{R}$) all the half-lines $\{x \in R \;:\; x \geq a\}$, $a \in \mathbb{R}$, then we do not obtain a topological space: the first and third axiom of topological spaces hold, but the second one does not (e.g. for the collection of all half lines with positive endpoints).

1.36 example. Let X be a set. The *discrete topology* on X is $\tau = \mathcal{P}(X)$. Check that τ is a topology. The indiscrete topology, or *trivial topology* on X is $\tau = \{\emptyset, X\}$. Check that the trivial topology is also a topology.

1.37 exercise. Let $\{C_\alpha\}_{\alpha\in A}$ be an arbitrary family of closed sets in a topological space X. Prove that the intersection $\bigcap_{\alpha\in A} C_\alpha$ is still a closed set.

1.38 definition. The *closure* $\overline{A}$ of a set $A \subset X$ is the smallest closed set containing A, that is

$$\overline{A} = \bigcap \{C \;:\; A \subset C\,,\; C \text{ closed set}\}\,.$$

A set $A \subset X$ is called *dense* in X if $\overline{A} = X$. A set $A \subset X$ is called *nowhere dense* if $X \setminus A$ is dense. A point $x \in X$ is called an *accumulation point* (or a *limit point*) of a set $A \subset X$ if every neighborhood of x contains infinitely many points of A. A point $x \in A$ is called an *interior point of A* if there exists a neighborhood of x entirely contained in A. The set of all interior points of A is called the *interior of A*, and is denoted by A°.

1.39 EXERCISE. Prove that a set A is open if and only if $A = A^\circ$. Prove that a set A is closed if and only if $A = \overline{A}$.

1.40 DEFINITION. A topological space X is said to be *separable* if there exists a countable, dense subset $S \subset X$.

1.41 EXERCISE. Show that $\mathbb{R}$ with the usual Euclidean topology is a separable space. Show that $\mathbb{R}$ endowed with the discrete topology (every set is open) is not separable.

1.42 DEFINITION. A point $x \in X$ is called a *boundary point* of a set $A \subset X$ if it is neither an interior point of A nor it is an interior point of $A^c = X \setminus A$. The set of boundary points of A is called the *boundary of A* and is denoted by ∂A.

1.43 EXERCISE. For very set $A \subset X$, prove that $\overline{A} = A \cup \partial A$. Consequently, prove that a set $C \subset X$ is closed if and only if C contains its boundary.

We now introduce the concept of convergent sequence in a topological space.

1.44 DEFINITION. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to converge to $x \in X$ if for every open set $O \subset X$ containing x there exists $N \in \mathbb{N}$ such that $\{x_n\}_{n\geq N} \subset O$. Any such point x is said a *limit* for the sequence $\{x_n\}_n$.

1.45 EXERCISE.

- Let $X = \mathbb{R}$ with the discrete topology (all sets are open). Prove that any subset $S \subset \mathbb{R}$ has neither accumulation nor boundary points, prove that the closure (as well as the interior) of every set S is the S itself, prove that the sequence $1/n$ does not converge to 0.
- Let $X = \mathbb{R}$ with the trivial topology (only the empty set and $\mathbb{R}$ are open). Prove that every sequence in $\mathbb{R}$ is convergent to any arbitrary point $x \in \mathbb{R}$.

The latter example above shows in particular that limits may be not unique in a general topological space.

We now introduce the concept of continuity for a function between two topological spaces.

The topological definition of continuity is simpler and more natural than the ϵ, δ definition for metric spaces.

1.46 DEFINITION. Let (X, τ), (Y, σ) be topological spaces. A map $f : X \to Y$ is said to be *continuous* if $O \in \sigma$ implies $f^{-1}(O) \in \tau$ (pre-images of open sets are open). f is an *open map* if $O \in \tau$ implies $f(O) \in \sigma$ (images of open sets are open). f is continuous at a point $x \in X$ if for any neighborhood A of $f(x)$ in Y, the pre-image $f^{-1}(A)$ contains a neighborhood of x in X.

1.47 exercise. Prove that a function f is continuous if and only if it is continuous at every point.

1.5 *The topology of metric spaces*

The concepts of convergence of a sequence in a metric space (introduced in Definition 1.16) and of continuous function between two metric spaces (introduced in Definition 1.23) can be formulated without the use of topologies and open sets. On the other hand, once the open sets are known, these two concepts are very naturally defined in a topological space, as seen in Definitions 1.44 and 1.46. Hence, in order to unify those concepts, we need to provide a standard way to equip all metric spaces with a topology.

Let (X, d) be a metric space. The *open ball*, $B_r(a)$, with radius $r > 0$ and center $a \in X$ is the set

$$B_r(a) \doteq \{x \in X \mid d(x, a) < r\}.$$

The *closed ball* $\overline{B}_r(a)$, is the set

$$\overline{B}_r(a) \doteq \{x \in X \mid d(x, a) \leq r\}.$$

1.48 exercise. Let $X = \mathbb{R}$ and let d_1 and d_2 the two distances on $\mathbb{R}$ defined in the example 1.18. Find $B_{1/2}(0)$.

1.49 definition. A subset G of a metric space X is *open* if for every $x \in G$ there is an $r > 0$ such that $B_r(x)$ is contained in G.

1.50 exercise. Let X be a metric space. Prove that

- the empty set $\emptyset$ and the whole space X are open,
- a finite intersection of open sets is open,
- an arbitrary union of open sets is open.

We prove here the second statement. Let $A_1, \ldots, A_n$ be open sets. Let $A = A_1 \cap \ldots \cap A_n$. In order to prove that A is open we must provide, for a given $x \in A$, a positive ϵ such that $B_\epsilon(x) \subset A$. Now, $x \in A$ means $x \in A_i$ for all $i = 1, \ldots, n$, and since each of those sets is an open set, there is an $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subset A_i$. Let

$$\epsilon = \min \{\epsilon_1, \ldots, \epsilon_n\}.$$

Clearly, $B_\epsilon(x) \subset B_{\epsilon_i}(x) \subset A_i$ for all i, and therefore $B_\epsilon(x) \subset A$.

The example below clarifies that the above reasoning may fail in case we are dealing with infinitely many open sets.

1.51 example. The interval $(-1/n, 1)$ is open in $\mathbb{R}$ for every $n \in \mathbb{N}$, but the intersection

$$\bigcap_{n=1}^{+\infty} I_n = [0, 1)$$

is not open (please, spend some time in proving the above identity as an exercise in case you are not convinced about it). Thus, an infinite intersection of open sets need not be open.

As a consequence of the above exercise 1.50, the family of open sets in a metric space X according to Definition 1.49 *is a topology*. With such an identification, all concepts we defined for topological spaces can be formulated for metric spaces. More precisely, *every metric space can be considered as a topological space, the topology being the one defined in Definition 1.49.* As a consequence, we can define *closed sets* in a metric space as all sets of the form $X \setminus G$ with G open.

First of all, we need to check that the concepts of convergence and continuity we provided for metric and topological spaces independently coincide on metric spaces.

1.52 exercise. Let x_n be a sequence in a metric space and let $x \in X$. Prove that x_n converges to x in the sense of Definition 1.16 if and only if x_n converges to x in the sense of Definition 1.44, with X a topological spaces in the sense of Definition 1.49.

1.53 exercise. Let X and Y be two topological spaces and let $f : X \to Y$ be a function. Prove that f is continuous in the sense of definition 1.46 if and only if it is continuous in the sense of Definition 1.23.

Closed sets in a metric spaces can be given an alternative, sequential characterisation as sets that contain their limit points.

1.54 proposition. *A subset F of a metric space is closed if and only if every convergent sequence in X with elements in F converges to a limit in F. That is, if $x_n \to x$ and $x_n \in F$ for all n, then $x \in F$.*

Proof. Assume first that F is closed, and let $x_n \in F$ with $x_n \to x$. Assume by contradiction that $x \in F^c$. Since F is closed, then F^c is open. Hence, there is an open ball $B_r(x)$ contained in F^c. This implies that no elements of the sequence x_n are contained in $B_r(x)$, and this contradicts the fact that x_n converges to x. Indeed, for $\epsilon = r$ there is no $n \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$.

Assume now that every convergent sequence in X with elements in F converges to a limit in F. We want to prove that F is closed. Assume by contradiction that F is not closed. This means that F^c is not open. Therefore, there exists $x \in F^c$ such that every open ball $B_\epsilon(x)$ intersects F. In particular, for all $n \in \mathbb{N}$ there is $x_n \in F$ such that $d(x, x_n) < 1/n$. Clearly x_n converges to x, but x does not belong to F, which contradicts the starting assumption. $\square$

1.55 exercise. Prove that a subset of a complete metric space is complete if and only if it is closed.

The closure of a set A can be also obtained by adding to A all limits of convergent sequences of elements of A. That is,

$$\overline{A} = \{x \in X : \text{ there is a sequence } (a_n) \subset A \text{ such that } a_n \to x\}. \tag{8}$$

It follows from (8) that A is a dense subset of the metric space X if and only if for every $x \in X$ there is a sequence (a_n) in A such that $a_n \to x$. Thus, every point in X can be approximated arbitrarily closely by points in the dense set A. We will encounter many dense sets later on.

This property has repercussions also on the concept of *separable* topological space that we introduced in Definition 1.40. A metric space is said to be separable if it is separable as a topological space, with the usual topology introduced at the beginning of this subsection.

For example, $\mathbb{R}$ with the usual standard metric is separable because $\mathbb{Q}$ is a countable dense subset. To see this, it suffices to write an arbitrary real number in decimal form and set zero on all the decimal digits from the $(n+1)$-th onward. This defines an approximating sequence in $\mathbb{Q}$.

1.56 example. The metric considered on a given set is crucial to determine whether or not a subset is dense. For example, consider $\mathbb{R}$ with the *discrete distance*

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

It is a trivial exercise to check that d is a distance. Now,

$$\begin{gathered}\text{a sequence } (x_n) \text{ in } \mathbb{R} \text{ converges to } x \text{ in the discrete distance} \\ \text{if and only if there exists } N \in \mathbb{N} \text{ such that } x_n = x \text{ for all } n \geq N.\end{gathered} \tag{9}$$

To see this, we apply the definition of convergence with $\epsilon = 1/2$, that is a sequence x_n converges to x if there is $N \in \mathbb{N}$ such that $d(x, x_n) < 1/2$ for all $n \geq N$. But the definition of discrete distance implies that $d(x, x_n)$ can only be less than $1/2$ if it is zero, that is if $x_n = x$. This proves (9). Now, it is quite clear that $\mathbb{Q}$ cannot be dense in $\mathbb{R}$ with such a distance. Indeed, the only subset of $\mathbb{R}$ which is dense in $\mathbb{R}$ equipped with the discrete distance is $\mathbb{R}$ *itself*. If $A \subset \mathbb{R}$ with $A \neq \mathbb{R}$, let $x \in \mathbb{R} \setminus A$, assuming by contradiction that (x_n) is a sequence in A converging to x, the fact (9) implies $x_n = x$ for all n larger than some $N \in \mathbb{N}$, but this is impossible since x is not in A and x_n is in A. As a consequence, $\mathbb{R}$ is not separable when equipped with the discrete distance.

According to Definition 1.49, a set U in a metric space X is a neighborhood of x if U contains a ball $B_r(x)$ centered at x for some $r > 0$. Definition 1.16 for the convergence of a sequence can therefore be rephrased in the following way. A sequence (x_n) converges to x if for every neighborhood U of x there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

1.6 *Compactness*

Compactness is one of the most important concepts in analysis. A simple and useful way to define compact sets in a metric space is by means of sequences. We first recall the concept of subsequence of a sequence in a metric space.

1.57 definition. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in the metric space (X, d). A subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ is a sequence

$$\mathbb{N} \ni k \mapsto x_{n_k}$$

such that the map

$$\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$$

is strictly increasing.

1.58 definition. A subset K of a metric space X is *sequentially compact* if every sequence in K has a convergent subsequence whose limit belongs to K.

We can take $K = X$ in this definition, so that a metric space X is sequentially compact if every sequence in X has a convergent subsequence. A subset K of (X, d) is sequentially compact if and only if the metric subspace (K, d_K) is sequentially compact.

1.59 example. The space of real numbers $\mathbb{R}$ is not sequentially compact. For example, the sequence (x_n) with $x_n = n$ has no convergent subsequence because $|x_n - x_m| \geq 1$ for all $m \neq n$. The closed, bounded interval $[0, 1]$ is sequentially compact, as we prove below. The half-open interval $(0, 1]$ is not a sequentially compact subset of $\mathbb{R}$, because the sequence $(1/n)$ converges to 0, and therefore has not subsequence with limit in $(0, 1]$. The limit does, however, belong to $[0, 1]$.

The full importance of compact sets will become clear only in the setting of infinite-dimensional normed spaces. It is nevertheless interesting to start with the finite-dimensional case. Compact subsets of $\mathbb{R}^n$ have a simple, explicit characterisation.

1.60 theorem (Heine-Borel). *A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

The fact that closed, bounded sets of $\mathbb{R}^n$ are sequentially compact is a consequence of the following, well-known theorem, called *Bolzano-Weierstrass theorem*.

1.61 theorem (Bolzano-Weierstrass). *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Compactness may be rephrased in ways that do not involve sequences. In fact, compactness is a *topological property*. We explain that in what follows.

Let A be a subset of a topological space X. We say that a collection $\{G_\alpha : \alpha \in \mathcal{A}\}$ of subsets of X is a *cover* of A if its union contains A, meaning that

$$A \subset \bigcup_{\alpha \in \mathcal{A}} G_\alpha.$$

We stress that the number of sets in the cover is not required to be countable. Indeed, the set of indexes $\mathcal{A}$ may have an arbitrary cardinality. If every G_α in the cover is open, then we say that $\{G_\alpha\}$ is an *open cover* of A.

Let $\epsilon > 0$. A subset $\{x_\alpha : \alpha \in \mathcal{A}\}$ of X is called an *ϵ-net* of the subset A if the family of open balls $\{B_\epsilon(x_\alpha) : \alpha \in \mathcal{A}\}$ is an open cover of $\mathcal{A}$. If the set $\{x_\alpha\}$ is finite, then we say that $\{x_\alpha\}$ is a *finite ϵ-net* of A.

1.62 definition (Total boundedness). A subset of a metric space is *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$.

That is, a subset A of a metric space X is totally bounded if for every $\epsilon > 0$ there is a finite set of points $\{x_1, x_2, \dots, x_n\}$ in X such that $A \subset \bigcup_{i=1}^n B_\epsilon(x_i)$.

We say that a cover $\{G_\alpha\}$ of A has a finite subcover if there is a finite subcollection of sets $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ such that $A \subset \bigcup_{i=1}^n G_{\alpha_i}$.

1.63 definition (Compactness). A subset K of a metric space X is *compact* if every open cover of K has a finite subcover.

1.64 example. The space of real numbers $\mathbb{R}$ is not compact, since the open cover $\{(n-1, n+1) : n \in \mathbb{Z}\}$ of $\mathbb{R}$ has no finite subcover. The half-open interval $(0,1]$ is not compact, since the open cover $\{(1/2n, 2/n) : n \in \mathbb{N}\}$ has no finite subcover. If this open cover is extended to an open cover of $[0,1]$, then the extension must contain an open neighborhood of 0. This open neighborhood, together with a finite number of sets from the previous cover of $(0,1]$, is a finite subcover of $[0,1]$, which is not surprising, since $[0,1]$ is indeed compact.

1.65 exercise. Prove that every totally bounded subset of a metric space is bounded.

The next Theorem is of paramount importance in that it establishes three equivalent formulations for compactness.

1.66 theorem. *Let (X, d) be a metric space, let $K \subset X$. Then, the following three conditions are equivalent:*

- *K is compact*
- *K is sequentially compact*
- *K is totally bounded and complete*

Proof. Assume K is compact. If K is not sequentially compact, there is a sequence $\{x_n\}_n$ in K without convergent subsequences. Hence, every $x \in K$ is not the limit of a subsequence of $\{x_n\}_n$. Hence, for all $x \in K$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x)$ contains at most finitely many elements of the sequence $\{x_n\}_n$. Since K is compact, the open cover $\{B_{\varepsilon_x}(x) : x \in K\}$ has a finite subcover $B_{\varepsilon_1}(x_1), \ldots, B_{\varepsilon_N}(x_N)$. As each of the (finitely) many balls above contains only finitely element of the sequence, which is a contradiction.

Assume now K is sequentially compact. Given a Cauchy sequence in K, compactness implies that there is a convergent subsequence. On the other hand, all convergent subsequences must have the same limit because the sequence is Cauchy, and therefore the sequence converges. Hence, K is complete. Assume now that K is not totally bounded. Then, there exists $\varepsilon > 0$ such that no finite balls with radius $\varepsilon > 0$ cover K. In particular, given one ball $B_\varepsilon(x_1)$ with $x_1 \in K$, there exists $x_2 \in K \setminus B_\varepsilon(x_1)$. Similarly, $B_\varepsilon(x_1)$ and $B_\varepsilon(x_2)$ do not cover K, hence there exists $x_3 \in K \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$. Inductively, we build a sequence $x_n \in K$ with the property $x_n \in K \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$, which implies $d(x_h, x_k) \geq \varepsilon$ whenever $h \neq k$. Consequently, such a sequence cannot have a convergent subsequence, which contradict the assumption of K being sequentially compact.

Assume now that K is totally bounded and complete. Assume K is not compact. Hence, let $\{U_\alpha : \alpha \in A\}$ be an open cover of K with no finite subcover. Without restriction, we remove from $\{U_\alpha\}$ all sets with empty intersection with K. Since K is totally bounded, there exists a finite $1/2$-net $x_1^1, \ldots, x_{m_1}^1$ in K. At least one of the balls $B_{1/2}(x_{j_1}^1)$ for some $j_1 \in \{1, \ldots, m_1\}$ is not covered by finitely many U_α's, otherwise clearly $\{U_\alpha\}$ would have a finite subcover. Hence, $A_1 := K \cap B_{1/2}(x_{j_1}^1)$ is not covered by finitely many U_α's. Once again, since K is totally bounded, there exists a finite $1/4$-net $x_1^2, \ldots, x_{m_2}^2$ in K. At least one of the balls $B_{1/4}(x_{j_2})$ for some $j_2 \in \{1, \ldots, m_2\}$ is not covered by finitely many U_α's and has a non empty intersection with A_1, otherwise, since the set of balls $B_{1/4}(x_j)$ with non empty intersection with A_1 covers A_1, if they all could be covered by finitely many U_α's then so would A_1, which is a contradiction. We set $A_2 := K \cap B_{1/4}(x_j)$. Inductively, we consider a 2^{-n}-net $x_1^n, \ldots, x_{m_n}^n$ in K and obtain the existence of a point $x_{j_n}^n$ which is the center of open ball of radius 2^{-n} with non empty intersection with $A_{n-1} := B_{2^{-(n-1)}}(x_{j_{n-1}}^{n-1}) \cap K$ and such that $B_{2^{-n}}(x_{j_n}^n)$ is not covered by finitely many U_α's. The sequence of centers $\{x_{j_n}^n\}_n$ is a Cauchy sequence in K. Indeed, since $B_{2^{-n}}(x_{j_n}^n) \cap B_{2^{-(n-1)}}(x_{j_{n-1}}^{n-1}) \neq \emptyset$, we get

$$d(x_{j_{n-1}}^{n-1}, x_{j_n}^n) \leq \frac{1}{2^{(n-1)}} + \frac{1}{2^n} \leq \frac{1}{2^{(n-2)}},$$

which implies, for $k < n$,

$$d(x_{j_k}^k, x_{j_n}^n) \leq 2^{-(n-2)} + \ldots + 2^{-(k-1)} \leq 2^{-(k-2)}.$$

Since K is complete, $x_{j_n}^n$ converges to some $x \in K$. Since K is covered by the U_α's, let $\alpha_0 \in A$ be such that $x \in U_{\alpha_0}$. As U_{α_0} is open, let $\varepsilon > 0$ be such that

$B_\varepsilon(x) \subset U_{\alpha_0}$. By the convergence of the sequence, let n be such that

$$d(x^n_{j_n}, x) < \varepsilon/2$$

and, without restriction, such that $2^{-n} < \varepsilon/2$. This implies $B_{2^{-n}}(x^n_{j_n}) \subset B_\varepsilon(x) \subset U_{\alpha_0}$. Hence, one of the balls $B_{2^{-n}}(x^n_{j_n})$ we constructed is covered by *one* of the sets of our given open cover, which is a contradiction. □

1.67 LEMMA. *Let (X, d) be a metric space and let $A \subset X$. Prove that A is dense if and only if for all $x \in X$ and for all $\varepsilon > 0$ there exists $a \in A$ such that $d(x, a) < \varepsilon$.*

Proof. Assume A is dense, and let $x \in X$. Assume by contradiction that there exists $\varepsilon > 0$ such that no $a \in X$ with $d(x, a) < \varepsilon$ belongs to A. As a consequence $B_\varepsilon(x) \subset X \setminus A$, which implies that no sequences in A can converge to x (otherwise infinitely many elements of the sequence would be in $B_\varepsilon(x)$, a contradiction). Assume now that for all $x \in X$ and for all $\varepsilon > 0$ there exists $a \in A$ such that $d(x, a) < \varepsilon$. We aim at proving that A is dense, that is $\overline{A} = X$, which is equivalent to $X \subset \overline{A}$. Let $x \in X$. Due to our hypothesis, we choose $\varepsilon = 1/n$ and get the existence of a point $a_n \in A$ with $d(x, a_n) < 1/n$. Hence, a_n is a sequence in A that converges to x, which means that $x \in \overline{A}$. □

The above Lemma may be used to prove the following Lemma.

1.68 LEMMA. *A sequentially compact metric space is separable.*

Proof. By theorem 1.66, there is a finite $(1/n)$-net A_n of a sequentially compact space K for every $n \in \mathbb{N}$. Let $A = \bigcup_{n=1}^{+\infty} A_n$ is countable[4] Moreover, A is dense in K by using the exercise 1.67. □

1.69 LEMMA. *Every compact subset K of a metric space X is closed and bounded.*

Proof. Using Proposition 1.54, let $x_n \in K$ be a sequence in K with limit $x \in X$. K is compact, and hence sequentially compact, due to Theorem 1.66, therefore $x \in K$ as it is the limit of all subsequences. Let us now assume that K is not bounded. Let $x_1 \in K$. For every $n \in \mathbb{N}$ there exists $x_2 \in K$ with $d(x_1, x_2) > 1$ as K is not bounded. Then, for the same reason there exists $x_3 \in K$ such that $d(x_1, x_3) > 2$. Inductively, there exists a sequence $x_n \in K$ such that $d(x_1, x_n) > n$. Hence, the family of open balls $B_n(x_n)$ clearly covers K but they have no finite subcover, which contradicts compactness. □

In the future, we will abbreviate 'sequentially compact' to 'compact' when referring to metric spaces. The following terminology if often convenient.

1.70 DEFINITION. A subset A of a metric space X is *precompact* if its closure in X is compact.

The term *relatively compact* is frequently used instead of 'precompact'. This definition means that A is precompact if every sequence in A has a convergent

[4]Exercise: prove that the countable union of finite sets is countable.

subsequence. The limit of the subsequence can be any point in X, and is not required to belong to A. Since compact sets are closed, a set is compact if and only if it is closed and precompact. A subset of a complete metric space is precompact if and only if it is totally bounded.

1.71 example. A subset of $\mathbb{R}^n$ is precompact if and only if it is bounded.

Continuous functions on compact sets have several nice properties. From proposition 1.28, continuous functions preserve the convergence of sequences. It follows immediately from definition 1.58 that continuous functions preserve compactness.

1.72 theorem. *Let $f : K \to Y$ be continuous on K, where K is a compact metric space and Y is any metric space. Then $f(K)$ is compact in Y.*

Since compact sets are bounded, continuous functions on a compact sets are bounded. Moreover, continuous functions on compact sets are uniformly continuous.

1.73 theorem. *Let $f : K \to Y$ be a continuous function on a compact set K. Then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then there is an $\epsilon > 0$ such that for all $\delta > 0$ there are $x, y \in K$ with $d(x, y) < \delta$ and $d(f(x), f(y)) \geq \epsilon$. Taking $\delta = 1/n$ for $n \in \mathbb{N}$, we find that there are sequences (x_n) and (y_n) in K such that

$$d(x_n, y_n) < \frac{1}{n}, \qquad d(f(x_n), f(y_n)) \geq \epsilon\,. \tag{10}$$

Since K is compact there are convergent subsequences of (x_n) and (y_n) which, for simplicity, we again denote by (x_n) and (y_n). From (10), the subsequences converge to the same limit, but the sequences $(f(x_n))$ and $(f(y_n))$ do not converge to the same limit. This contradicts the continuity of f. □

We conclude this subsection with a Theorem that is of paramount importance in topology and analysis.

1.7 *Maxima and minima*

As highlighted in fact 0.7 and example 0.8, maximum and minimum problems are of central importance in applications. The mathematical formulation of these problems is the maximisation or minimisation of a real-valued function f on a state space X. Each point of the state space, which is often a metric space, represents a possible state of the system. The existence of a maximising, or minimising, point of f in X may not be at all clear; indeed, such a point may not exist. The following theorem gives sufficient conditions for the existence of maximising or minimising points - namely, that the function f is continuous and the state space X is compact. Although these conditions are fundamental, they are too strong to be useful in many applications. We will return to these issues later on.

1.74 theorem. *Let K be a compact metric space and $f : K \to \mathbb{R}$ a continuous real-valued function. Then, f is bounded on K and attains its maximum and minimum. That is, there are points $x, y \in X$ such that*

$$f(x) = \inf_{z \in K} f(z) \qquad f(y) = \sup_{z \in K} f(z).$$

Proof. From theorem 1.72, the image $f(K)$ is a compact subset of $\mathbb{R}$, and therefore f is bounded by the Heine-Borel theorem 1.60. It is enough to prove that f attains its minimum, because the application of the result to $-f$ implies that f attains its maximum. Since f is bounded, it is bounded from below, and the infimum m of f on K is finite. By the definition of the infimum, for each $n \in \mathbb{N}$ there is an $x_n \in K$ such that

$$m \leq f(x_n) < m + \frac{1}{n}.$$

This inequality implies that

$$\lim_{n \to +\infty} f(x_n) = m. \tag{11}$$

The sequence (x_n) need not converge, but since K is compact the sequence has a convergent subsequence, which we denote by (x_{n_k}). We denote the limit of the subsequence by x. Then, since f is continuous, we have from (11) that

$$f(x) = \lim_{k \to +\infty} f(x_{n_k}) = m.$$

Therefore, f attains its infimum m at x. □

The strategy of this proof is typical of many compactness arguments. We construct a sequence of approximate solutions of our problem, in this case a minimising sequence (x_n) that satisfies (11). We use compactness to extract a convergent subsequence, and show that the limit of the convergent subsequence is a solution of our problem, in this case a point where f attains its infimum.

1.8 *Exercises*

1. Let $(X, \|\cdot\|)$ be a linear normed space on $\mathbb{R}$ (or $\mathbb{C}$), and let d be the induced metric on X, i. e. $d(x, y) = \|x - y\|$. Prove that

   - d is translation invariant, i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$,
   - d is 1-homogeneous, i.e. $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

2. Let $(X, \|\cdot\|)$ be a normed linear space. For $x, y \in X$ define

$$d(x,y) \doteq \frac{\|x-y\|}{1+\|x-y\|}.$$

- Prove that (X, d) is a metric space.
- Prove that, for all $x, y, z \in X$, one has the translation invariance property

$$d(x+z, y+z) = d(x,y).$$

3. Prove that the map $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} &d((x_1,y_1),(x_2,y_2)) \\ &= \begin{cases} 1 & \text{if } \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \geq 1 \\ \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} & \text{if } \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} < 1 \end{cases} \end{aligned}$$

is a metric on $\mathbb{R}^2$

4. Let the sequence of rational numbers (x_n) be defined recursively via the formula

$$x_{n+1} = \frac{x_n^2+2}{2x_n}, \qquad n = 1,2,3,\ldots, \qquad x_1 = 2.$$

(a) Prove that $x_n \geq \sqrt{2}$ for all $n \geq 1$.

(b) Prove that $x_{n+1} \leq x_n$ for all $n \geq 1$, i.e. the sequence is *increasing* in n (Hint: use (a)).

(c) Prove that (x_n) is convergent (Hint: use some basic theory of sequences, monotone sequences... boundedness...).

(d) Prove that (x_n) is a Cauchy sequence (Hint: estimate directly the difference $x_n - x_{n+1}$).

(e) Prove that the limit of (x_n) is irrational.

(f) Use the above to prove that not all Cauchy sequences of rational numbers are convergent in $\mathbb{Q}$.

5. Let X be a set, and let d be the distance

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Let x_n be a convergent sequence in the metric space (X, d). Prove that there exists $N \in \mathbb{N}$ such that $x_n = x_m$ for all $n, m \geq N$, i.e. prove that the sequence is eventually constant.

6. Let A be a subset of a metric space (X, d). Prove that A is bounded (i.e. $\operatorname{diam} A$ is finite) if and only if there exists $x_0 \in X$ and $M \in \mathbb{R}$ such that $d(x_0, x) \leq M$ for all $x \in A$.

7. Let $s : \mathbb{R} \to \mathbb{R}$ defined by $s(x) = x^2$. Prove that s is continuous on $\mathbb{R}$ but not uniformly. Prove that $s|_{[a,b]}$ is uniformly continuous for al $a, b \in \mathbb{R}$, $a < b$.

8. Prove that an affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $f(x) = Ax + b$, where A is a constant $m \times n$ matrix and b is a constant m-vector.

9. Prove that an affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous.

10. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Prove that the Cartesian product $Z = X \times Y$ is a metric space with the metric d defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

11. Let X be a normed linear space. A series $\sum_{n=1}^{+\infty} x_n$ in X is *absolutely convergent* if $\sum_{n=1}^{+\infty} \|x_n\|$ converges to a finite value in $\mathbb{R}$. Prove that X is a Banach space if and only if every absolutely convergent series converges.

12. Let $f : \mathbb{R} \to \mathbb{R}$, with $\mathbb{R}$ equipped with the usual Euclidean distance. Let

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0. \end{cases}$$

Prove that f is lower semi-continuous.

13. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Show that the composition $h = g \circ f : X \to Z$ defined by $h(x) = g(f(x))$ is also continuous.

14. Suppose that F and G are closed and open subsets of $\mathbb{R}^n$, respectively, such that $F \subset G$. Show that there is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that

   - $0 \leq f \leq 1$
   - $f(x) = 1$ for all $x \in F$,
   - $f(x) = 0$ for all $x \in G^c$.

   Hint: consider the function

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, f)}.$$

   This result is called (a special case of) *Uhryson's lemma*.

15. Prove that a closed subset of a compact space is compact.

16. Let (X, d) be a metric space and let $Y \subset X$. Prove that (Y, d) is complete if and only if Y is a closed subset of X.

2 Spaces of continuous functions

In section 1, we introduced the notion of normed linear space, with finite dimensional Euclidean space $\mathbb{R}^n$ as the main example. In this section, we study linear spaces of continuous functions on a compact set equipped with the uniform norm. These function spaces are our first examples of infinite-dimensional normed linear spaces, and we explore the concepts of convergence, completeness, density, and compactness in this context. More practically, we learn for the first time how to compute distances between functions. Functions will be treated as points in a linear space equipped with a norm. We will focus in particular on the problem of *compactness* of sets of functions. As an application, we prove an existence result for initial value problems for ordinary differential equations.

2.1 *Convergence of function sequences*

Suppose that (f_n) is a sequence of real-valued functions $f_n : X \to \mathbb{R}$ defined on a metric space X. What should we mean by $f_n \to f$? Two natural ways to answer this question are the following:

- The functions f_n are defined by their real values $f_n(x) \in \mathbb{R}$ with x varying in X. So the sequence of functions converges if the values $x_n(x)$ converge. That is, we say $f_n \to f$ if $f_n(x) \to f(x)$ for all $x \in X$. This definition reduces the convergence of a function sequence to the convergence of real numbers, which which we are already familiar. This type of convergence is called *pointwise convergence*.
- We define a suitable notion of the distance between functions, and say that $f_n \to f$ id the distance between f_n and f tends to zero. In this approach, we regard the functions as points in a metric space, and use *metric convergence*.

Both of these ideas are useful. It turns out, however, that they are not compatible. For most domains X - for example any uncountable domain - pointwise convergence cannot be expressed as convergence with respect to a metric. The next example shows that pointwise convergence is not a good notion of convergence to use for continuous functions because it does not preserve continuity.

2.1 example. We define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = x^n.$$

It is easily checked that the sequence (f_n) converges pointwise to the function f given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

The pointwise limit f is discontinuous at $x = 1$.

In view of these somewhat pathological features of pointwise convergence, we consider metric convergence. As we will see, there are many different ways to define a distance between functions, and different metrics or norms usually lead to different types of convergence. A natural norm on spaces of continuous functions the *uniform* or *sup norm*, which is defined by

$$\|f\|_\infty \doteq \sup_{x\in X} |f(x)|. \tag{12}$$

The norm $\|\cdot\|_\infty$ is finite if and only if f is bounded. The reason for the notation will become clear when we study L^p spaces later on. Two functions are close in the metric associated with the uniform norm if their pointwise values are uniformly close. Metric convergence with respect to the uniform norm is called *uniform convergence*.

2.2 definition. A sequence of bounded, real-valued functions (f_n) on a metric space X converges uniformly to a function f is

$$\lim_{n\to+\infty} \|f_n - f\|_\infty = 0.$$

Uniform convergence implies pointwise convergence. This fact is an easy exercise that we leave to the student. The sequence defined in example 2.1 shows that the opposite implication does not hold, since $f_n \to f$ pointwise, but $\|f_n - f\|_\infty = 1$ for every n, and hence $\|f_n - f\|_\infty$ does not converge to zero. Unlike pointwise convergence, uniform convergence preserves continuity.

2.3 theorem. *Let (f_n) be a sequence of bounded, continuous, real-valued functions on a metric space (X, d). If $f_n \to f$ uniformly, then f is continuous.*

Proof. In order to show that f is continuous at $x \in X$, we need to prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. By the triangle inequality, we have

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_n(y)|.$$

Since $f_n \to f$ uniformly, there is an $n \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{\epsilon}{3}, \qquad |f(y) - f_n(y)| < \frac{\epsilon}{3} \qquad \text{for all } x, y \in X.$$

Since f_n is continuous ar x, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3}.$$

It follows that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$, so f is continuous at x. □

The '$\epsilon/3$-trick' used in this proof has many other applications. The proof fails if $f_n \to f$ pointwise but not uniformly.

The notion of uniform convergence of a sequence of functions can be easily extended to *series of functions*. Given a sequence of functions (f_n) on a metric space X, consider the sequence of partial sums

$$S_n(x) = \sum_{k=1}^{n} f_n(x).$$

We say that the series $\sum f_n$ converges pointwise at $x \in X$ if the series of real numbers $\sum f_n(x)$ converges. We say that $\sum f_n$ converges uniformly if the sequence of functions (S_n) converges uniformly.

2.2 *Spaces of continuous functions*

Let X be a metric space. We denote the set of continuous, real-valued functions $f : X \to \mathbb{R}$ by $C(X)$. The set $C(X)$ is a real linear space under the pointwise addition and functions and the scalar multiplication of functions by real numbers. That is, for $f, g \in C(X)$ and $\lambda \in \mathbb{R}$, we define

$$(f+g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda f(x).$$

From theorem 1.74, a continuous function f on a compact metric space K is bounded, so the uniform norm $\|f\|_\infty$ is finite for $f \in C(K)$. It is straightforward to check that $C(K)$ equipped with the uniform norm is a normed linear space. For example, the triangle inequality holds because

$$\|f+g\|_\infty = \sup_{x\in K} |f(x)+g(x)| \le \sup_{x\in K} |f(x)| + \sup_{x\in K} |g(x)| = \|f\|_\infty + \|g\|_\infty.$$

We will always use the uniform norm on $C(K)$, unless we state explicitly otherwise. A basic property of $C(K)$ is that it is complete, and therefore a Banach space.

2.4 theorem. *Let K be a compact metric space. The space $C(K)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence in $C(K)$ with respect to the uniform norm. We have to show that (f_n) converges uniformly. We do this in two steps. First we construct a candidate function f for the limit if the sequence, second we show that the sequence converges uniformly to f.

The fact that (f_n) is Cauchy implies that the sequence of real numbers $(f_n(x))$ is Cauchy in $\mathbb{R}$ for each $x \in K$. Indeed,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty,$$

and the latter term is less than $\epsilon > 0$ for n, m larger than some $N = N(\epsilon)$. Since $\mathbb{R}$ is complete, the sequence of pointwise values converges, and we can define a function $f : K \to \mathbb{R}$ by

$$f(x) = \lim_{n\to+\infty} f_n(x).$$

For the second step, we use the fact that (f_n) is Cauchy in $C(K)$ to prove that it converges uniformly to f. Since $f_m(x) \to f(x)$ as $m \to +\infty$, we have

$$\begin{aligned} \|f_n - f\|_\infty &= \sup_{x\in K} |f_n(x) - f(x)| = \sup_{x\in K} \lim_{m\to+\infty} |f_n(x) - f_m(x)| \\ &\le \liminf_{m\to+\infty} \sup_{x\in K} |f_n(x) - f_m(x)|. \end{aligned} \tag{13}$$

The last inequality above uses elementary properties of lim inf and sup proven in example 1.22. The fact that (f_n) is Cauchy in the uniform norm means that for all $\epsilon > 0$ there is an N such that

$$\sup_{x\in K} |f_n(x) - f_m(x)| < \epsilon \qquad \text{for all } n, m \ge N.$$

It follows from (13) that $\|f_n - f\|_\infty \le \epsilon$ for all $n \ge N$, which proves that $\|f_n - f\|_\infty \to 0$ as $n \to +\infty$. By theorem 2.3, the limit function f is continuous, and therefore belongs to $C(K)$. Hence, $C(K)$ is complete. □

2.5 example. Suppose $K = \{x_1, \dots, x_n\}$ is a finite space, with metric d defined by $d(x_i, x_j) = 1$ for $i \ne j$. A function $f : K \to \mathbb{R}$ can be identified with a point $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, where $f(x_j) = y_j$, and

$$\|f\|_\infty = \max_{1\le i\le n} |y_i|.$$

The space $C(K)$ is linearly isomorphic to the finite-dimensional space $\mathbb{R}^n$ with the maximum norm, which we have already observed is a Banach space. If K contains infinitely many points, for example if $K = [0, 1]$, then $C(K)$ is an infinite-dimensional Banach space.

The same proof applies to complex-valued functions $f : K \to \mathbb{C}$, and the space of complex-valued continuous functions on a compact metric space is also a Banach space with the uniform norm.

The pointwise product of two continuous functions is continuous, so $C(K)$ has an *algebra structure*. The product is compatible with the norm, in the sense that

$$\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty. \tag{14}$$

We way that $C(K)$ is a Banach algebra. Strict inequality may occur in (14); for example, the product of two functions that are nonzero on disjoint sets is zero.

Equation (12) does not define a norm on $C(K)$ when X is not compact, since continuous functions may be unbounded. The space $C_b(X)$ of bounded continuous functions on X is a Banach space with respect to the uniform norm.

2.6 definition. The *support*, supp$f$, of a function $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$) on a metric space X is the closure of the set on which f is nonzero,

$$\operatorname{supp} f \doteq \overline{\{x \in X : f(x) \ne 0\}}.$$

We say that f has compact support if supp is a compact subset of X, and denote the space of continuous functions on X with compact support by $C_c(X)$.

The space $C_c(X)$ is a linear subspace of $C_b(X)$, but it need not be closed, in which case it is not a Banach space. We denote the closure of $C_c(X)$ in $C_b(X)$ by $C_0(X)$. Since $C_0(X)$ is a closed linear subspace of a Banach space, it is also a Banach space. We have the following inclusions between these spaces of continuous functions:

$$C(X) \supset C_b(X) \supset C_0(X) \supset C_c(X).$$

If X is compact, then these spaces are equal.

2.7 example. A function $f : \mathbb{R}^d \to \mathbb{R}$ has compact support if there is an $R > 0$ such that $f(x) = 0$ for all x with $\|x\| > R$. The space $C_0(\mathbb{R}^d)$ consists of continuous functions that vanish at infinity, meaning that for every $\epsilon > 0$ there is an $R > 0$ such that $\|x\| > R$ implies that $|f(x)| < \epsilon$. We write this condition as $\lim_{\|x\|\to+\infty} f(x) = 0$.

2.8 example. Consider real functions $f : \mathbb{R} \to \mathbb{R}$. then $f(x) = x^2$ is in $C(\mathbb{R})$ but not in $C_b(\mathbb{R})$. The constant function $f(x) = 1$ is in $C_b(\mathbb{R})$ but not $C_0(\mathbb{R})$. The function $f(x) = e^{-x^2}$ is in $C_0(\mathbb{R})$ but not in $C_c(\mathbb{R})$. The function

$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

is in $C_c(\mathbb{R})$.

2.3 *Approximation by polynomials*

A *polynomial* $p : [a, b] \to \mathbb{R}$ on a closed, bounded interval $[a, b]$ is a function of the form

$$p(x) = \sum_{k=0}^{n} c_k x^k,$$

where the coefficients c_k are real numbers. If $c_n \neq 0$, the integer $n \geq 0$ is called the *degree* of p. Clearly, polynomials are a special case of continuous functions. The Weierstrass Approximation Theorem states that every continuous function $f : [a, b] \to \mathbb{R}$ can be approximated by a polynomial with arbitrary accuracy in the uniform norm.

2.9 theorem (Weierstrass approximation). *The set of polynomials on $[a, b]$ is dense in $C([a, b])$.*

Proof. We need to show that for any $f \in C([a, b])$ there is a sequence of polynomials (p_n) such that $p_n \to f$ uniformly.

We first show that, by shifting and rescaling x, it is sufficient to prove the assertion in the case $[a,b] = [0,1]$. We define $T : C([a,b]) \to C([0,1])$ by

$$(Tf)(x) = f(a + (b-a)x).$$

Then, T is a linear invertible map, with inverse

$$(T^{-1}f)(x) = f\left(\frac{x-a}{b-a}\right).$$

Moreover, T is an isometry (see the Exercises), since $\|Tf\|_\infty = \|f\|_\infty$, and for any polynomial p both Tp and $T^{-1}p$ are polynomials. If polynomials are dense in $C([0,1])$, then for any $f \in C([a,b])$ we have polynomials p_n such that $p_n \to Tf$ in $(C([0,1]), \|\cdot\|_\infty)$. Since T^{-1} is continuous (see the Exercises), the polynomials $T^{-1}p_n$ converge to f in $C([a,b])$.

To show that polynomials are dense in $C([0,1])$, we use a proof by Bernstein, which gives an explicit formula for a sequence of polynomials converging to a function f in $C([0,1])$. These polynomials are called the Bernstein polynomials of f, and are defined by

$$B_n(x; f) \doteq \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

It is an easy exercise to show that each term $x^k(1-x)^{n-k}$ attains its maximum at $x = k/n$. Here,

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

The binomial theorem implies that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^k = 1.$$

Therefore, the difference between f and its n-th Bernstein polynomial can be written as

$$B_n(x; f) - f(x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] \binom{n}{k} x^k (1-x)^{n-k}.$$

Taking the supremum with respect to x of the absolute value of this equation, we get

$$\|B_n(\cdot; f) - f(x)\|_\infty \leq \sup_{0 \leq x \leq 1} \left[\sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \right]. \quad (15)$$

Here, we use $B_n(x; f)$ to denote the value of the Bernstein polynomial at x, and $B_n(\cdot; f)$ to denote the corresponding polynomial function.

Let $\epsilon > 0$ be an arbitrary positive number. From Theorem 1.73, the function

f is uniformly continuous, so there is a δ such that

$$|x-y|<\delta \quad \text{implies} \qquad |f(x)-f(y)|<\epsilon,$$

for all $x, y \in [0,1]$. To estimate the right-hand-side of (15), we divide the terms in the series into two groups. We let

$$\begin{aligned} I(x) &= \{k : \ 0 \le k \le n \text{ and } |x-(k/n)|<\delta\}, \\ J(x) &= \{k : \ 0 \le k \le n \text{ and } |x-(k/n)| \ge \delta\}. \end{aligned}$$

Then, we get the following estimate,

$$\begin{aligned} \|B_n(\cdot; f) - f\|_\infty \le & \epsilon \sup_{0\le x\le 1} \left[\sum_{k\in I(x)} \binom{n}{k} x^k (1-x)^{n-k} \right] \\ & + \sup_{0\le x\le 1} \left[\sum_{k\in J(x)} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \right] \\ & \le \epsilon + 2\|f\|_\infty \sup_{0\le x\le 1} \left[\sum_{k\in J(x)} \binom{n}{k} x^k (1-x)^{n-k} \right]. \end{aligned} \tag{16}$$

Since $[x-(k/n)]^2 \ge \delta^2$ for $k \in J(x)$, the sum on the last term in (16) can be estimated as follows:

$$\begin{aligned} & \sup_{0\le x\le 1} \left[\sum_{k\in J(x)} \binom{n}{k} x^k (1-x)^{n-k} \right] \\ & \quad \le \frac{1}{\delta^2} \sup_{0\le x\le 1} \left[\sum_{k\in J(x)} \left(x - \frac{k}{n} \right)^2 \binom{n}{k} x^k (1-x)^{n-k} \right] \\ & \quad \le \frac{1}{\delta^2} \sup_{0\le x\le 1} \left[\sum_{k=0}^{n} \left(x^2 - \frac{2k}{n} + \frac{k^2}{n^2} \right) \binom{n}{k} x^k (1-x)^{n-k} \right] \\ & \quad = \frac{1}{\delta^2} \sup_{0\le x\le 1} \left[x^2 B_n(x; 1) - 2x B_n(x; x) + B_n(x; x^2) \right]. \end{aligned} \tag{17}$$

To find an expression for the Bernstein polynomials $B_n(x; 1)$, $B_n(x; x)$, and $B_n(x; x^2)$, we write out the binomial expansion of $(x+y)^n$, compute the first and second derivatives of the expansion with respect to x, and rearrange the results. This gives

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$
$$x(x+y)^{n-1} = \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} x^k y^{n-k},$$
$$\left(\frac{n-1}{n}\right) x^2 (x+y)^{n-2} + \left(\frac{1}{n}\right) x(x+y)^{n-1} = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k y^{n-k}.$$

Evaluation of these equations at $y = 1 - x$ gives

$$B_n(x;\,1) = 1$$
$$B_n(x;\,x) = x$$
$$B_n(x;\,x^2) = \left(\frac{n-1}{n}\right) x^2 + \left(\frac{1}{n}\right) x,$$

for all $n \geq 1$. Inserting these terms in (17), after some manipulations we easily obtain from (16)

$$\|B_n(\cdot;\,f) - f\|_\infty \leq \epsilon + \frac{\|f\|_\infty}{2n\delta^2}.$$

Taking the lim sup of the above inequality as $n \to +\infty$, we get

$$\limsup_{n \to +\infty} \|B_n(\cdot;\,f) - f\|_\infty \leq \epsilon.$$

Since ϵ is arbitrary, we have that $B_n(\cdot;\,f)$ converge uniformly to f.

□

The Weierstrass approximation theorem differs from Taylor's theorem, which states that a function with sufficiently many derivatives can be approximated locally by its Taylor polynomial. The Weierstrass approximation theorem applied to a continuous function, which need not be differentiable, and states that there is a global polynomial approximation of the function on the whole interval $[a, b]$.

2.4 *Compact subsets of* $C(K)$

The proof of the Heine-Borel theorem, that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded, uses the finite-dimensionality of $\mathbb{R}^n$ in an essential way. Compact subsets of infinite-dimensional normed spaces are also closed and bounded, but these properties are no longer sufficient for compactness. In this subsection, we prove the Arzelà-Ascoli theorem, which characterises the compact subsets of $C(K)$.

2.10 definition. Let $\mathcal{F}$ be a family of functions from a metric space (X, d_X) to a metric space (Y, d_Y). The family $\mathcal{F}$ is *equicontinuous* if for every $x \in X$ and

$\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $f \in \mathcal{F}$.

The crucial point in this definition is that δ does not depend on f, although it may depend on x. If δ can be chosen independent of x as well, then the family is said to be *uniformly equicontinuous*. The following theorem is a generalisation of theorem 1.73. The proof is left as an exercise (see the Exercises).

2.11 theorem. *An equicontinuous family of functions from a compact metric space to a metric space is uniformly equicontinuous.*

Next, we give necessary and sufficient conditions for compactness in $C(K)$.

2.12 theorem (Arzelá-Ascoli). *Let K be a compact metric space. A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.*

Proof. Recall that a set is precompact if its closure is compact, and that a set is compact if and only if it is closed and precompact.

Step 1. We first prove that a precompact subset is equicontinuous. Suppose $\mathcal{F}$ is a precompact subset of $C(K)$. Fix $\epsilon > 0$. Since $\mathcal{F}$ is dense in $\overline{\mathcal{F}}$, we have

$$\overline{\mathcal{F}} \subset \bigcup_{f \in \mathcal{F}} B_{\epsilon/3}(f).$$

Since $\overline{\mathcal{F}}$ is compact, there is a finite subset $\{f_1, \dots, f_k\}$ of $\mathcal{F}$ such that

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{k} B_{\epsilon/3}(f_i).$$

Each f_i is uniformly continuous by Theorem 2.11, so there is a $\delta_i > 0$ such that $d(x,y) < \delta_i$ implies $|f_i(x) - f_i(y)| < \epsilon/3$ for all $x, y \in K$. We define δ by

$$\delta \doteq \min_{1 \le i \le k} \delta_i.$$

Clearly, $\delta > 0$. For every $f \in \mathcal{F}$, there is an $1 \le i \le k$ such that $\|f - f_i\|_\infty < \epsilon/3$. We conclude that for $d(x,y) < \delta$

$$|f(x) - f(y)| \le |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \epsilon.$$

Since ϵ is arbitrary and δ is independent of f, the set $\mathcal{F}$ is equicontinuous.

Step 2. Assume $\mathcal{F}$ is bounded and equicontinuous. We want to show that $\mathcal{F}$ is pre-compact. Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ be a sequence in $\mathcal{F}$. It suffices to prove that f_n has a subsequence which is a Cauchy sequence. This will then imply that the subsequence will converge to some $f \in C(X)$, as $C(X)$ is a complete metric space. Let $\epsilon > 0$, and let $\delta > 0$ given as by the equicontinuity assumption. Since K is compact, we can cover K with finitely many balls $B_\delta(x_1), \dots, B_\delta(x_N)$ with radius δ (K is totally bounded). Hence, for a given $x \in K$, $x \in B_\delta(x_j)$ for some $j \in \{1, \dots, N\}$. Hence, we can use the equicontinuity and get

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| + |f_m(x_j) - f_m(x)| \\ &\leq 2\epsilon + |f_n(x_j) - f_m(x_j)|.\end{aligned}$$

Now, the sequence $v_n := (f_n(x_1), \ldots, f_n(x_N)) \in \mathbb{R}^N$ is bounded in view of $\mathcal{F}$ being bounded. Therefore, in view of Heine-Borel theorem there exists a convergent subsequence v_{n_k}. This means that there exists $M \in \mathbb{N}$ such that for all $h, k \geq M$ we have

$$|f_{n_h}(x_j) - f_{n_k}(x_j)| < \epsilon, \qquad \text{for all } j \in \{1, \ldots, N\}.$$

Therefore, $|f_{n_h}(x) - f_{n_k}(x)| < 3\epsilon$ for all $x \in X$. We have therefore shown that, for a given $\epsilon > 0$ we can extract a subsequence $\{f_n^{(1)}\}$ such that

$$\|f_n^{(1)} - f_m^{(1)}\|_\infty \leq 3\epsilon, \qquad \text{for all } n, m \geq M_\epsilon,$$

for a suitable $M_\epsilon \in \mathbb{N}$. Now, fix $\epsilon = 1$ and extract the subsequence $\{f_n^{(1)}\}$ as above. Then fixe $\epsilon = 1/2$ and extract from $\{f_n^{(1)}\}$ another subsequence $\{f_n^{(2)}\}$, and so on. The diagonal sequence $\{f_n^{(n)}\}$ will satisfy

$$\|f_n^{(n)} - f_m^{(m)}\|_\infty \leq \frac{3}{k}$$

for $n, m \geq M_k$ for some suitable M_k depending on k. Hence, $f_n^{(n)}$ is a Cauchy sequence, and the assertion follows. ☐

2.13 example. For each $n \in \mathbb{N}$, we define a function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 2^{-n} \\ 2^{n+1}(x - 2^{-n}) & \text{if } 2^{-n} \leq x \leq 3 \cdot 2^{-(n+1)} \\ 2^{n+1}(2^{-(n-1)} - x) & \text{if } 3 \cdot 2^{-(n+1)} \leq x \leq 2^{-(n-1)} \\ 0 & \text{if } 2^{-(n-1)} \leq x \leq 1. \end{cases}$$

These functions consist of 'tent' functions of height one that move from right to left across the interval $[0,1]$, becoming narrower and steeper as they do so.

Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. Then $\|f_n\|_\infty = 1$ for all $n \geq 1$, so $\mathcal{F}$ is bounded, but $\|f_m - f_n\|_\infty = 1$ for all $m \neq n$, so the sequence (f_n) does not have any convergent subsequence. Hence, the set $\mathcal{F}$ is a closed, bounded subset of $C([0,1])$ which is not compact. Note that $\mathcal{F}$ is not equicontinuous either, because the graphs of f_n become steeper as n gets larger.

2.14 definition. A function $f : X \to \mathbb{R}$ on a metric space X is *Lipschitz continuous* on X if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C d(x,y) \qquad \text{for all } x, y \in X. \tag{18}$$

We will often abbreviate the term 'Lipschitz continuous' to 'Lipschitz'.

2.15 exercise. Prove that every Lipschitz continuous function is uniformly continuous.

2.16 example. The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous, but it is not Lipschitz, because

$$\lim_{x \to 0^+} \frac{|g(x) - g(0)|}{|x - 0|} = +\infty.$$

If $f : X \to \mathbb{R}$ is a Lipschitz function, then we define the *Lipschitz constant* $\mathrm{Lip}(f)$ of f by

$$\mathrm{Lip}(f) \doteq \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Equivalently, $\mathrm{Lip}(f)$ is the smallest constant C that works in the Lipschitz condition (18), i.e.

$$\mathrm{Lip}(f) = \inf\{C : |f(x) - f(y)| \leq C d(x,y) \text{ for all } x, y \in X\}.$$

Suppose that K is a compact metric space and $M > 0$. We define a subset $\mathcal{F}_M$ of $C(K)$ by

$$\mathcal{F}_M = \{f : f \text{ is Lipschitz on } K \text{ and } \mathrm{Lip}(f) \leq M\}.$$

The set $\mathcal{F}_M$ is equicontinuous, since if $\epsilon > 0$ and $\delta = \epsilon/M$, then

$$d(x,y) < \delta \quad \text{implies} \quad |f(x) - f(y)| < \epsilon \quad \text{for all } f \in \mathcal{F}_M.$$

The set $\mathcal{F}_M$ is closed, since if (f_n) is a sequence in $\mathcal{F}_M$ that converges uniformly to f in $C(K)$, then

$$\begin{aligned}
\mathrm{Lip}(f) &= \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} \\
&= \sup_{x \neq y} \left[\lim_{n \to +\infty} \frac{|f_n(x) - f_n(y)|}{d(x,y)} \right] \\
&\leq \liminf_{n \to +\infty} \left[\sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{d(x,y)} \right] \\
&\leq M,
\end{aligned}$$

where we have use example 1.22. Thus, the limit f belongs to $\mathcal{F}_M$. The set $\mathcal{F}_M$ is not bounded, since the constant functions belong to $\mathcal{F}_M$ and their sup-norms are arbitrarily large. Consequently, although $\mathcal{F}_M$ itself is not compact, the Arzelà-Ascoli theorem implies that every closed, bounded subset of $\mathcal{F}_M$ is compact, and every bounded subset of $\mathcal{F}_M$ is precompact.

2.17 EXAMPLE. Suppose that x_0 is a point in a compact metric space K. Let

$$\mathcal{B}_M = \{f \in \mathcal{F}_M \,:\, f(x_0) = 0\}.$$

Then, $\mathcal{B}_M$ is bounded because for every $f \in \mathcal{B}_M$ we have

$$\|f\| = \sup_{x \in K} |f(x) - f(x_0)| \le M \sup_{x \in K} |x - x_0| \le M\mathrm{diam}(K),$$

where $\mathrm{diam}(K)$ is finite since K is compact, and hence bounded. The set $\mathcal{B}_M$ is closed, since if $f_n(x_0) = 0$ and $f_n \to f$ in $C(K)$, then

$$f(x_0) = \lim_{n \to +\infty} f_n(x_0) = 0.$$

Therefore, the set $\mathcal{B}_M$ is a compact subset of $C(K)$.

A continuously differentiable function with bounded partial derivatives on a convex, open subset of $\mathbb{R}^n$ is Lipschitz, see the Exercises. A Lipschitz continuous function need not be differentiable everywhere, however, since its graph may have corners.

2.18 EXAMPLE. The absolute value function $f(x) = |x|$ is Lipschitz continuous with Lipschitz constant one, because

$$|f(x) - f(y)| = ||x| - |y|| \le |x - y|.$$

However, f is not differentiable at $x = 0$.

2.19 EXAMPLE. Let $C^1([0,1])$ denote the space of all continuous functions f on $[0,1]$ with continuous derivative f'. For constants $M > 0$ and $N > 0$, we define the subset $\mathcal{F}$ of $C([0,1])$ by

$$\mathcal{F} = \{f \in C^1([0,1]) \,:\, \|f\|_\infty \le M\,,\ \|f'\|_\infty \le N\}.$$

For all $f \in \mathcal{F}$ and $x, y \in [0,1]$ we have

$$|f(x) - f(y)| = \left| \int_x^y f'(z)dz \right| \le |x - y| \|f'\|_\infty \le M|x - y|.$$

Since $\mathcal{F}$ is bounded in $C([0,1])$, Arzelà-Ascoli theorem implies that $\mathcal{F}$ is precompact in $C([0,1])$. However, $\mathcal{F}$ is not closed, because the uniform limit of continuously differentiable functions need not be differentiable. Thus, $\mathcal{F}$ is not compact. Its closure in $C([0,1])$ is the compact set

$$\overline{\mathcal{F}} = \{f \in C([0,1]) \,:\, \|f\| \le M\,,\ \mathrm{Lip}(f) \le N\}.$$

A family of continuously differentiable functions with uniformly bounded derivatives is equicontinuous, see also the Exercises. If the family is also bounded, then it is precompact. The idea that a uniform bound on suitable norms of the derivatives of a family of functions implies that the family is precompact will reappear in the study of *Sobolev spaces*.

2.20 example. In many applications one may need to work with functions that are *continuously differentiable*, i.e. functions f such that the first derivative f' is continuous. Such a set is typically referred to as $C^1(A)$ where A is the domain of the functions. Let us consider the case $A = [a,b]$. A well known calculus theorem states that $C^1([a,b]) \subset C([a,b])$, that is, every continuously differentiable function is also continuous. Clearly, $C^1([a,b])$ is a linear space. This is a simple exercise (every linear combination of continuously differentiable functions is continuously differentiable). Hence, $C^1([a,b])$ may be seen as a linear subspace of $C([a,b])$. Then, a natural question arises: is $C^1([a,b])$ closed in $(C([a,b]), \|\cdot\|_\infty)$?

The answer to said question is *no, it is not*. Indeed, it is not difficult to construct examples of sequences of C^1 functions converging uniformly to a function that is not C^1. For instance, consider $[a,b] = [-1,1]$ and

$$f_n(x) = \frac{nx^2}{1+n|x|}, \qquad x \in [-1,1].$$

We claim that $\|f_n - f\|_\infty \to 0$ as $n \to +\infty$ with $f(x) = |x|$. Indeed, computing

$$|f_n(x) - f(x)| = \frac{|nx^2 - |x|(1+n|x|)|}{1+n|x|} = \frac{|x|}{1+n|x|},$$

and observing that $|f_n - f|$ is therefore an even function, we have

$$\|f_n - f\|_\infty = \max_{x\in[0,1]} \frac{x}{1+nx}.$$

By computing the derivative

$$\frac{d}{dx}\frac{x}{1+nx} = \frac{1+nx-nx}{(1+nx)^2} = \frac{1}{(1+nx)^2} \geq 0$$

we decuce the maximum above is achieved at $x = 1$, therefore $\|f_n - f\|_\infty = \frac{1}{1+n} \to 0$, which proves the uniform convergence. On the other hand, it is well known that the function $f(x) = |x|$ is not differentiable at $x = 0$.

Since $C^1([a,b])$ is in general not closed in $C([a,b])$, we automatically deduce that $(C^1([a,b]), \|\cdot\|_\infty)$, is not a Banach space (it is a normed linear space indeed, but it is not complete!). In order to make $C^1([a,b])$ a complete normed linear space we need to equip it with the norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty.$$

Such a norm incorporates the information on the first derivative as well.

Let us prove that $(C^1([a,b]), \|\cdot\|_{C^1})$ is complete. Assume $f_n \in C^1([a,b])$ is a Cauchy sequence. Since

$$\|f_n - f_m\|_\infty \leq \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty = \|f_n - f_m\|_{C^1}$$

we deduce that f_n is Cauchy in the space $(C([a,b]), \|\cdot\|_\infty)$, which is a Banach space. Therefore, there exists $f \in C([a,b])$ such that $f_n \to f$ in the $\|\cdot\|_\infty$ norm. Similarly, we deduce that f_n' is Cauchy in $(C([a,b]), \|\cdot\|_\infty)$ and for the same reason there exists $g \in C([a,b])$ such that $f_n' \to g$ in $\|\cdot\|_\infty$. Now, if we prove that $g = f'$ then we would get that

$$\|f_n - f\|_\infty + \|f_n' - f'\|_\infty \to 0$$

which would prove the assertion. On the other hand, since f_n is continuously differentiable, from the fundamental theorem of integral calculus we get

$$f_n(x) = \int_a^x f_n'(y)dy.$$

By letting $n \to +\infty$ on both sides of the above identity, we get

$$f(x) = \int_a^x g(y)dy,$$

because on the right hand side $f_n(x)$ converges pointwise to $f(x)$ (uniform implies pointwise) and on the left hand side we know that we can interchange limit and integral if the convergence is uniform (f_n' converges uniformly to f). Hence, the fundamental theorem of integral calculus implies once again that $f' = g$.

2.21 example. The above example can be extended as follows to functions with k-derivatives. The space $C^k([a,b])$ of k-times continuously differentiable functions on $[a,b]$ is not a Banach space with respect to the sup-norm $\|\cdot\|_\infty$ for $k \geq 1$, since the uniform limit of continuously differentiable functions need not be differentiable. We define the C^k-norm by

$$\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty + \ldots + \|f^{(k)}\|_\infty.$$

Then $C^k([a,b])$ is a Banach space with respect to the C^k-norm. Convergence with respect to the C^k-norm is uniform convergence of functions and their first k derivatives. We omit the details.

2.5 *Application to differential equations*

As outlined in the introduction to this course, functional analysis provides tools to solving mathematical problems in an abstract setting. Differential equations are a major example in this setting, either the case of ordinary differential equations (ODEs) and the case of partial differential equations (PDEs). In the remaining part of this subsection we shall use functional analysis to prove a classical existence theorem for differential equations called *Peano's theorem*.

We consider a scalar, first order ODE for a real-valued function $u(t)$ of the form

$$\dot{u} = f(t, u). \tag{19}$$

In (19), we use $\dot{u}(t)$ to denote the derivative of u with respect to t, and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given *continuous* function. A solution of (19), defined in an open interval $I \subset \mathbb{R}$, is a continuously differentiable function $j : I \to \mathbb{R}$ such that

$$\dot{u}(t) = f(t, u(t)) \qquad \text{for all } t \in I.$$

If the solution is defined on the whole $\mathbb{R}$, then we call it a *global solution*. If the solution is defined only on a subinterval of $\mathbb{R}$, the we call it a *local solution*. We will refer to the independent variable t as *time*. We consider the *initial value problem* (IVP)

$$\begin{cases} \dot{u}(t) = f(t, u) \\ u(t_0) = u_0. \end{cases} \tag{20}$$

Here $t_0 \in \mathbb{R}$ is a given initial time, and u_0 is a given initial value. It is known that the only assumption of f being continuous does not imply that (20) has a *unique* solution, see the Exercises. However, the continuity of f guarantees the *existence* of at least one solution.

2.22 theorem (Peano). *Suppose that $f(t, u)$ is a continuous function on $\mathbb{R}^2$. Then, for every $(t_0, u_0) \in \mathbb{R}^2$ there is an open interval $I \subset \mathbb{R}$ that contains t_0, and a continuously differentiable function $u : I \to \mathbb{R}$ that satisfies the initial value problem* (20).

Proof. [5]

We say that $u_\epsilon(t)$ is an *ϵ-approximate solution* of (20) in an interval I containing t_0 if:

(a) $u_\epsilon(t_0) = u_0$,

(b) $u_\epsilon(t)$ is a continuous function of t that is differentiable at all but finitely many points of I,

(c) at every point $t \in I$ where $\dot{u}_\epsilon(t)$ exists, we have

$$|\dot{u}_\epsilon(t) - f(t, u_\epsilon(t))| < \epsilon.$$

To construct an ϵ-approximate solution u_ϵ, we first pick $T_1 > 0$, and let

$$I_1 \doteq \{t \,:\, |t - t_0| \leq T_1\}.$$

We partition I_1 into $2N$ subintervals of length h, where $T_1 = Nh$, and let

$$t_k \doteq t_0 + kh \qquad \text{for } -N \leq k \leq N.$$

[5] The proof is not part of the exam

We denote the values of the approximate solution at the times t_k by $u_\epsilon(t_k) \doteq a_k$. For $0 \le k \le N$, we define these values by the following finite difference approximation of the ODE,

$$\frac{a_{k+1} - a_k}{h} = f(t_k, a_k),$$
$$a_0 = u_0.$$

This discretisation of (20) is called the forward Euler method. It is not an accurate numerical method for the solution of (20), but its simplicity makes it convenient for an existence proof. For $-N \le k \le 0$ we use the backward Euler method

$$\frac{a_k - a_{k-1}}{h} = f(t_k, a_k),$$
$$a_0 = u_0.$$

Inside the subinterval $t_k \le t \le t_{k+1}$, we define $u_\epsilon(t)$ to be the linear function of t that takes the appropriate values at the endpoints. That is, for $0 \le k \le N$ we set

$$u_\epsilon(t) = a_k + b_k(t - t_k) \qquad \text{for } t_k \le t \le t_{k+1},$$

where the parameters a_k and b_k are defined recursively by

$$a_0 = u_0, \qquad a_k = a_{k-1} + hb_{k-1},$$
$$b_0 = f(t_0, u_0), \qquad b_k = f(t_k, a_k).$$

For $-N \le k \le -1$, we set

$$u_\epsilon(t) = a_k + b_{k+1}(t - t_k) \qquad \text{for } t_k \le t \le t_{k+1},$$

with

$$a_0 = u_0, \qquad a_k = a_{k+1} - hb_{k+1},$$
$$b_0 = f(t_0, u_0), \qquad b_k = f(t_k, a_k).$$

Thus, $u_\epsilon(t)$ is a continuous, piecewise linear function of t that is differentiable except possibly at the points $t = t_k$, and $\dot{u}_\epsilon(t) = b_k$ for $t_k < t < t_{k+1}$. For $t_k < t < t_{k+1}$ with $k \ge 0$, we have

$$|\dot{u}_\epsilon(t) - f(t, u_\epsilon(t))| = |f(t_k, a_k) - f(t, a_k + b_k(t - t_k))|, \tag{21}$$
$$|t - t_k| \le h, \qquad |a_k + b_k(t - t_k) - a_k| \le |b_k|h. \tag{22}$$

A similar estimate can be carried out for negative k:

$$|\dot{u}_\epsilon(t) - f(t, u_\epsilon(t))| = |f(t_{k+1}, a_{k+1}) - f(t, a_k + b_{k+1}(t - t_k))|, \tag{23}$$
$$|t - t_{k+1}| \le h, \qquad |a_k + b_{k+1}(t - t_k) - a_{k+1}| \le 2|b_{k+1}|h. \tag{24}$$

We choose an $L > 0$, and a $T \leq T_1$ such that the graph of every u_ϵ with $|t - t_0| \leq T$ is contained in the rectangle $R \subset \mathbb{R}^2$ given by

$$R = \{(t, u) : |t - t_0| \leq T, |u - u_0| \leq L\}.$$

To do this, we consider the closed rectangle $R_1 \subset \mathbb{R}^2$, centered at (t_0, u_0), defined by

$$R_1 = \{(t, u) : |t - t_0| \leq T_1, |u - u_0| \leq L\}.$$

We let

$$M \doteq \sup\{|f(t, u)| : (t, u) \in R_1\}, \qquad T \doteq \min(T_1, L/M).$$

It follow that, for $|t - t_0| \leq T$, the slopes b_k of the linear segments of u_ϵ are less than or equal to M, and the graph of u_ϵ lies in the cone bounded by the lines $u - u_0 = M(t - t_0)$ and $u - u_0 = -M(t - t_0)$.

Since R is compact, the function f is uniformly continuous on R. Therefore, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(s, u) - f(t, v)| \leq \epsilon$$

for all $(s, v), (t, v) \in R$ such that $|s - t| \leq \delta$ and $|u - v| \leq \delta$. Using (21), (22), (23), and (24), we see that u_ϵ is an ϵ-approximate solution when $h \leq \delta$ and $2Mh \leq \delta$.

Each u_ϵ is Lipschitz continuous, and its Lipschitz constant is bounded uniformly by M, independently of ϵ. We also have $u_\epsilon(t_0) = u_0$ for all ϵ. From example 2.17, the set $\{u_\epsilon\}$ is precompact in $C([t_0 - T, t_0 + T])$. Hence, there is a continuous function u and a sequence (ϵ_n) with $\epsilon_n \to 0$ as $n \to +\infty$ such that $u_{\epsilon_n} \to u$ as $n \to +\infty$ uniformly on $[t_0 - T, t_0 + T]$.

It remains to show that the limiting function u solves (20). Since u_ϵ is piecewise linear, we have

$$\begin{aligned} u_\epsilon(t) &= u_\epsilon(t_0) + \int_{t_0}^{t} \dot{u}_\epsilon(s)ds \\ &= u_0 + \int_{t_0}^{t} f(s, u_\epsilon(s))ds + \int_{t_0}^{t} [\dot{u}_\epsilon(s) - f(s, u_\epsilon(s))]ds. \end{aligned} \tag{25}$$

Here, $\dot{u}_\epsilon$ is not necessarily defined at the points t_k, but this does not affect the value of the integral. We set $\epsilon = \epsilon_n$ in (25) and let $n \to +\infty$. Since f is uniformly continuous, for a given $\sigma > 0$ we can find a $\mu > 0$ such that $|u - v| < \mu$ implies $|f(t, u) - f(t, v)| < \sigma$ for all $t \in [t_0 - T, t_0 + T]$. Since u_{ϵ_n} converges uniformly to u in $[t_0 - T, t_0 + T]$, there is an $N \in \mathbb{N}$ such that $\sup_{t \in [t_0 - T, t_0 + T]} |u_{\epsilon_n}(t) - u(t)| < \mu$ for all $n \geq N$, and consequently

$$\sup_{t \in [t_0 - T, t_0 + T]} |f(t, u_{\epsilon_n}(t)) - f(t, u(t))| < \sigma$$

for all $n \geq N$, where it is clear that N only depends on σ. This shows that $f(\cdot, u_{\epsilon_n}(\cdot)) \to f(\cdot, u(\cdot))$ uniformly in $[t_0 - T, t_0 + T]$. Hence,

$$\left| \int_{t_0}^{t} f(s, u_{\epsilon_n}(s))ds - \int_{t_0}^{t} f(s, u(s))ds \right| \leq \int_{t_0}^{t} |f(s, u_{\epsilon_n}(s)) - f(s, u(s))|ds$$
$$\leq T \sup_{t \in [t_0 - T, t_0 + T]} |f(t, u_{\epsilon_n}(t)) - f(t, u(t))| \to 0, \qquad \text{as } n \to +\infty.$$

Moreover, since u_{ϵ_n} is an ϵ_n-approximate solution, we have the trivial estimate

$$\left| \int_{t_0}^{t} [\dot{u}_\epsilon(s) - f(s, u_\epsilon(s))]ds \right| \leq |t - t_0|\epsilon_n \leq T\epsilon_n \to 0, \qquad \text{as } n \to +\infty.$$

Hence, (25) with $\epsilon = \epsilon_n$ and $n \to +\infty$ reads

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds. \tag{26}$$

The fundamental theorem of calculus implies that the right hand side of (26) is continuously differentiable. Therefore, u is also continuously differentiable in $|t - t_0| \leq T$, and $\dot{u}(t) = f(t, u(t))$. □

More generally, the same proof applies if f is continuous only in some open set $D \subset \mathbb{R}^2$ which contains the initial point (t_0, u_0), provided we choose the rectangles R_1 and R so that they are contained in D.

Apart from revealing a crucial application of Arzelà's theorem in a subject - namely differential equations - strongly linked with applications in science and engineering, the strategy used in the proof of Peano's theorem is pedagogically relevant in that it shows how to use functional analysis to approximate an infinite dimensional object (the solution to an ODE, i.e. a function on a real interval) by a finite dimensional one, namely the vector

$$(a_{-N}, a_{-N+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{N-1}, a_N).$$

This very simple idea is at the basis of numerical calculus, as the student will see later on in her/his studies.

2.6 *The contraction mapping theorem and its applications*

In this subsection we state and prove that *contraction mapping theorem*, which is one of the simplest and most useful methods for the construction of linear and nonlinear equations. We also present a number of applications of the theorem.

2.23 definition. Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction mapping*, or *contraction*, if there exists a constant c, with $0 \leq c < 1$, such that

$$d(T(x), T(y)) \leq cd(x, y) \tag{27}$$

for all $x, y \in X$.

Thus, a contraction maps points closer together. In particular, for every $x \in X$, and any $r > 0$, all points y in the ball $B_r(x)$ are mapped into a ball $B_s(T(x))$, with $s < r$. Clearly, all contractions are Lipschitz continuous, and hence uniformly continuous.

If $T : X \to X$, then a point $x \in X$ such that

$$T(x) = x \tag{28}$$

is called a *fixed point* of T. The contraction mapping theorem states that a contraction on a complete metric space has a unique fixed point. The contraction mapping theorem is only one example of what are more generally called fixed-point theorems. For example, the *Schauder fixed point theorem* states that a continuous mapping on a convex, compact subset of a Banach space has a fixed point. We will not discuss the proof in this course.

In general, the condition that c is strictly less than one is needed for the uniqueness and existence of the fixed point. For example, if $X = \{0, 1\}$ is the discrete metrics space with metric determined by $d(0, 1) = 1$, then the map T defined by $T(0) = 1$, $T(1) = 0$ satisfies (27) with $c = 1$, but T does not have any fixed points. On the other hand, the identity map on any metric space satisfies (27) with $c = 1$, and every point is a fixed point. It is worth noting that (28), and hence its solutions, do not depend on the metric d. Thus, if we can find any metric on X such that X is complete and T is a contraction on X, then we obtain the existence and uniqueness of a fixed point.

2.24 theorem (Contraction mapping). *If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then T has exactly one fixed point.*

Proof. The proof is constructive, meaning that we will explicitly construct a sequence converging to the fixed point. Let $x_0 \in X$ be any point in X. We define a sequence (x_n) in X by

$$x_{n+1} = T(x_n), \qquad \text{for } n \geq 0.$$

To simplify the notation, we often omit the parentheses around the argument of a map. We denote the n-th iterate of T by T^n, so that $x_n = T^n x_0$.

First, we show that (x_n) is a Cauchy sequence. If $n \geq m \geq 1$, then from

(27) and the triangle inequality. we have

$$
\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\leq c^m d(T^{n-m} x_0, x_0) \\
&\leq c^m \left[d(T^{n-m} x_0, T^{n-m-1} x_0) + \ldots + d(T x_0, x_0) \right] \\
&\leq c^m \left[\sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0) \\
&\leq c^m \left[\sum_{k=0}^{+\infty} c^k \right] d(x_1, x_0) \\
&= \left(\frac{c^m}{1-c} \right) d(x_1, x_0),
\end{aligned}
$$

which implies that (x_n) is a Cauchy sequence since $0 \leq c < 1$. Since X is complete, (x_n) converges to a limit $x \in X$. By continuity of T, we get

$$
Tx = T\left(\lim_{n \to +\infty} x_n \right) = \lim_{n \to +\infty} T x_n = \lim_{n \to +\infty} x_{n+1} = x,
$$

which shows that x is a fixed point. Finally, let $x, y \in X$ be two fixed points, then

$$
0 \leq d(x, y) = d(Tx, Ty) \leq c d(x, y).
$$

Since $c < 1$, we have $d(x, y) = 0$, so $x = y$ and the fixed point is unique. □

In the Exercises we consider a simple application of the contraction mapping theorem in a finite dimensional case. Applications to spaces of infinite dimensions are more challenging. We start by considering the most classical one, namely the next uniqueness theorem for systems of ODEs.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, with notation $f = f(t, u)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, be a continuous function which is also Lipschitz continuous with respect to the u variable on a neighborhood of the point $(t_0, u_0) \in \mathbb{R}^{n+1}$. Consider the IVP

$$
\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0. \end{cases} \tag{29}
$$

The problem (29) can be reformulated as an integral equation

$$
u(t) = u_0 + \int_{t_0}^{t} f(s, u(s)) ds. \tag{30}
$$

By the fundamental theorem of calculus, a continuous solution of (30) is a continuously differentiable solution of (29). Equation (30) can be written as a fixed point equation

$$
u = Tu
$$

for the map T defined by

$$Tu(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds.$$

we want to find conditions which guarantee that T is a contraction on a suitable space of continuous functions.

2.25 THEOREM (Cauchy-Lipschitz). *Suppose $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, and let $t_0 \in I$. Suppose that f is a continuous function of (t, u), and a locally Lipschitz function of u around $u_0 \in \mathbb{R}^n$, uniformly in t, on $I \times \mathbb{R}^n$, i.e. suppose that there are two constant $C, R > 0$ such that*

$$|f(t,u) - f(t,v)| \leq C|u - v| \qquad \text{for all } u, v \in B_R(u_0) \text{ and all } t \in I. \tag{31}$$

Then, there are a subinterval $[t_0, t_0 + \delta) \subset I$ and a unique continuously differentiable function $u : [t_0, t_0 + \delta) \to \mathbb{R}^n$ that satisfies (29).

Proof. [6]

We will show that T is a contraction on the space of continuous functions defined on a time interval $t_0 \leq t \leq t_0 + \delta$ for small δ. First we show that T is well defined on a bounded subset of $C([t_0, t_0 + \delta))$. Assuming $\|u - u_0\|_\infty \leq r$ for some $r \in (0, R]$, we estimate

$$\begin{aligned}\|Tu - u_0\|_\infty &\leq \left\| \int_{t_0}^{t_0+\delta} \left(|f(s,u(s)) - f(s,u_0)| + |f(s,u_0)|\right) ds \right\|_\infty \\ &\leq \delta \left(C\|u - u_0\|_\infty + \|f(\cdot, u_0)\|_\infty\right) \leq \delta(Cr + +K),\end{aligned}$$

where $K = \sup_{t \in I} |f(t, u_0)|$. By choosing $\delta > 0$ small enough and $r > 0$ such that

$$\frac{\delta K}{1 - C\delta} < r,$$

we obtain $\|Tu - u_0\|_\infty \leq r$, and the operator T is well defined from $\overline{B_{u_0}(r)} \subset C([t_0, t_0 + \delta))$ into itself. Note that $\overline{B_{u_0}(r)}$ is a closed subset of a complete metric space and is therefore complete.

Now, suppose that $u, v : [t_0, t_0 + \delta) \to \mathbb{R}^n$ are two continuous functions.

[6] The proof is not part of the exam

Then, from (31), we estimate

$$\begin{aligned}
\|Tu - Tv\|_\infty &= \sup_{t\in[t_0,t_0+\delta)} |Tu(t) - Tv(t)| \\
&= \sup_{t\in[t_0,t_0+\delta)} \left| \int_{t_0}^{t_0+\delta} (f(s,u(s)) - f(s,v(s)))ds \right| \\
&\leq \sup_{t\in[t_0,t_0+\delta)} \int_{t_0}^{t_0+\delta} |(f(s,u(s)) - f(s,v(s)))|ds \\
&\leq C \sup_{t\in[t_0,t_0+\delta)} \int_{t_0}^{t_0+\delta} |u(s) - v(s)|ds \\
&\leq C\delta\|u - v\|_\infty.
\end{aligned}$$

It follow that if $\delta < 1/C$ then T is a contraction of $\overline{B_{u_0}(r)} \subset C([t_0, t_0 + \delta))$ into itself. Therefore there is a unique solution $u : [t_0, t_0 + \delta) \to \mathbb{R}^n$. □

2.7 *Exercises*

1. For the following sequences of functions on a specific domain $K \subset \mathbb{R}$, answer the following questions: 1) find (if it exists) a function f on the same domain K such that f_n converges pointwise to f; 2) say whether or not f_n converges fo f uniformly; motivate all the answers with suitable mathematical reasoning:
   - $f_n(x) = x^n$ on $K = [0,1]$,
   - $f_n(x) = nxe^{-nx}$ on $K = [0,1]$ and on $K = \mathbb{R}$,
   - $f_n(x) = (1-x)x^n$ on $K = [0,1]$,
   - $f_n(x) = \frac{nx}{1+n^2x^2}$ on $K = [-1,1]$,
   - $f_n(x) = \frac{n^2x^2}{1+n^2x^2}$ on $K = [-1,1]$,
   - $f_n(x) = \frac{x}{1+n^2x^2}$ on $K = \mathbb{R}$,
   - $f_n(x) = nxe^{-n^2x}$ on $K = [0,1]$,
   - $f_n(x) = \sin(x/n)$.
2. Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be two normed spaces. A *linear isometry* between A and B is a linear map $T : A \to B$ such that $\|Tx\|_B = \|x\|_A$. Prove that every surjective, linear isometry is also invertible. Prove that both T and T^{-1} are continuous.
3. Consider the Banach space $X = C([-1,1])$ equipped with the usual infinity norm $\|\cdot\|$. Say which of the following subsets of X are *closed* and which ones are *dense* in X. For those sets that are not closed, find the *closure of the set*.
   (a) $H = \{f \in X : f(0) = 0\}$

(b) $H = \{f \in X : f(x) = 0 \text{ for all } x \in [-1, 0]\}$
(c) $H = \{f \in X : f \text{ is a polynomial of degree} \leq 1\}$
(d) $H = \{f \in X : f \text{ is a polynomial}\}$
(e) $H = \{f \in X : f \text{ is a Lipschitz function}\}$
(f) $H = \{f \in X : f \text{ is a Lipschitz function with Lipschitz constant} \leq 1\}$
(g) $H = \{f \in X : f \text{ is even}\}$
(h) $H = \{f \in X : f \text{ is odd}\}$
(i) $H = \{f \in X : f \text{ is strictly increasing}\}$
(j) $H = \{f \in X : f \text{ has a local minimum at } x = 0\}$

4. Consider the Banach space $X = C([-1, 1])$ equipped with the usual infinity norm $\|\cdot\|$. Consider the set

$$H = \{f \in X : f \text{ is strictly decreasing}\}.$$

Find $\overline{H}$. Justify your answer.

5. Prove that an equicontinuous family of functions from a compact metric space to a metric space is uniformly equicontinuous.

6. For each $n \in \mathbb{N}$, consider $f_n(x) = \sin(n\pi x)$. Is the family of functions $\{f_n : n \in \mathbb{N}\}$ compact in $C([0, 1])$ equipped with the uniform norm? Motivate your answer.

7. Let $\mathcal{F}$ be the subset of $C([0, 1])$ that consists of functions f of the form

$$f(x) = \sum_{n=1}^{+\infty} a_n \sin(n\pi x) \qquad \text{with} \quad \sum_{n=1}^{+\infty} n|a_n| \leq 1.$$

(a) Prove that actually f is an element of $C([0, 1])$
(b) Prove that $\mathcal{F}$ is bounded in $C([0, 1])$
(c) Prove that $\mathcal{F}$ is precompact in $C([0, 1])$

8. Let $\{f_n\}_{n\in\mathbb{N}} \subset C([0, 1])$ such that $\sup_{x\in[0,1]} |f_n(x)| \leq 1$ for any $n \in \mathbb{N}$. Define $F_n : [0, 1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^x f_n(t)\, dt.$$

Show that the sequence $\{F_n\}_{n\in\mathbb{N}}$ has a subsequence that converges uniformly on $[0, 1]$.

9. Let G the following subset of $C([0, \pi])$

$$G = \left\{ g \in C([0, \pi]; g(x) = \int_0^\pi \sin(xy) f(y)\, dy,\ f \in C([0, \pi]),\ \|f\|_\infty \leq 1 \right\}.$$

Prove that G is relatively compact in in $C([0, \pi])$.

10. Suppose that $f : C \to \mathbb{R}$ is a continuously differentiable function on an open, convex subset C of $\mathbb{R}^n$, and that the partial derivatives of f are bounded on C. Prove that f is Lipschitz.

11. Prove that a family of continuously differentiable functions on an open, convex subset C of $\mathbb{R}^n$ with uniformly bounded partial derivatives is equicontinuous.

12. Give a counterexample to show that $f_n \to f$ in $C([0,1])$ and f_n continuously differentiable does not imply that f is continuously differentiable.

13. Consider the space of continuously differentiable functions,

$$C^1([(a,b)]) = \{f : [a,b] \to \mathbb{R} : \ f \text{ and } f' \text{ are continuous}\},$$

equipped with the C^1 norm,

$$\|f\|_{C^1} \doteq \|f\|_\infty + \|f'\|_\infty.$$

Prove that $C^1([(a,b)])$ is a Banach space.

14. Prove that the set of Lipschitz continuous functions on $[0,1]$ with Lipschitz constant less than or equal to one and zero integral is compact in $C([0,1])$.

15. Prove that $C([a,b])$ is separable.

16. Consider the *discrete dynamical system*

$$x_{n+1} = Tx_n\,, \qquad x_0 \in [0,1],$$

defined on $[0,1]$ with

$$Tx = 4\mu x(1-x).$$

Such a system describes the *logistic growth* of a population. Prove that if $0 \le \mu < 1/4$ there is a unique initial datum x_0 such that $x_n = x_0$ for all $n \in \mathbb{N}$. What happens if $\mu \ge 1/4$?

17. An infinite sum of functions $\sum_{n=1}^{+\infty} f_n(x)$ converges *totally* if the series $\sum_{n=1}^{+\infty} \|f_n\|_\infty$ is convergent. Prove that if $\sum_{n=1}^{+\infty} f_n(x)$ converges totally then $\sum_{n=1}^{+\infty} f_n(x)$ converges uniformly.

3 Measure and integration. L^p spaces

Using the sup norm is by far the simplest way to measure the distance between two functions. On the other hand, this distance has the downside of being sometime 'too strong'. In many applications, convergence with respect to weaker distance my be extremely useful, for examples based on *integral* quantities. For instance, given two functions f and g on the interval $[0,1]$, we may ask ourselves how much they differ in an integral sense by evaluating the quantity

$$\int_0^1 |f(x) - g(x)|^2 dx.$$

The above quantity (its square root, more precisely) has very likely all the properties of a distance. Defining said quantity relies on the concept of *integral* of a function of real variables. We all are familiar with *Riemann*'s integral, which therefore seems to be the natural candidate concept to use in this context. However, we will see very soon that there is a major theoretical issue with classical Riemann integration theory when we try to adapt it to the goals of functional analysis. Hence, in this section we will introduce the *Lebesgue* theory of integration, which will allow is to defined the so-called *L^p spaces*, also called *Lebesgue spaces*.

We all know that there is a close interplay between *integrals* of functions $f : \mathbb{R}^d \to \mathbb{R}$ and the way we *measure* sets in $\mathbb{R}^d$. For example, if we define the function $f : A \to \mathbb{R}$ as $f(x) \equiv 1$, with a given $A \subset \mathbb{R}^3$, we expect that $\int_A f(x)dx$ returns the 3-dimensional volume of the set A. Now, Riemann's theory of integrals relies on the so-called *Peano-Jordan* theory of measure, which we will briefly recall here. Then, we shall introduce *Lebesgue measure theory*, which is sort of a counterpart (on the measure side) of Lebesgue integral theory.

3.1 *Integrals and measures*

We all are familiar with *Riemann's integration theory*.

Given a bounded function $f : [a,b] \to \mathbb{R}$, Riemann's idea to compute the integral relies essentially on approximating the area of the subgraph of f (with the convention that regions in which f is negative give a contribution with negative sign) with piecewise constant functions, i.e. with functions which are constant on intervals. The integral $\int_a^b h(x)dx$ of a piecewise constant function h is trivially computed as the sum of areas of rectangles. We set

$$\underline{\int_a^b} f(x)dx \doteq \sup \left\{ \int_a^b h(x)dx \; : \; h \le f \text{ and } h \text{ is piecewise constant} \right\},$$

and

$$\overline{\int_a^b} f(x)dx \doteq \inf \left\{ \int_a^b h(x)dx \ : \ h \geq f \text{ and } h \text{ is piecewise constant} \right\}.$$

If $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$ we say that f is Riemann integrable, and define

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

The class of Riemann integrable functions contains e.g. continuous functions and piecewise continuous functions.

Riemann's integral can be extended also to functions defined on subsets of $\mathbb{R}^d$. We refer to [1].

Strictly related with integration is the way we *measure* sets. In one space dimension, the measure of a set is (roughly speaking) the length of the set, which reduces e.g. to $b - a$ in the case of an interval $[a, b]$. In two dimensions the measure of $A \subset \mathbb{R}^2$ is the area of A, in three dimensions the measure of $A \subset \mathbb{R}^3$ is the volume of A. The goal of measure theory is to define the measure of elementary sets (intervals in one dimension, rectangles in two dimensions, etc) and use them to define the measure of more and more complicated sets, e.g. open sets, closed sets, etc.

3.1 fact. Let $R \subset \mathbb{R}^n$ be a *rectangle*, i.e. R is the Cartesian product of n intervals of the form $[a, b)$, $(a, b]$, (a, b), $[a, b]$. The measure $m(R)$ of a rectangle can be easily computed as the produce of the sizes of said intervals. An *elementary set* $I \subset \mathbb{R}^n$ is the finite union of rectangles. Every elementary set can be written as the union of pairwise disjoint rectangles (easy exercise), i.e. $I = R_1 \cup \ldots \cup R_m$ with $R_i \cap R_j = \emptyset$ if $i \neq j$. Hence, the measure of the elementary set I above can be computed as

$$m(I) = \sum_{i=1}^{m} m(R_i).$$

A classical way to extend the notion of measure in $\mathbb{R}^n$ to more complicated sets leads to *Peano-Jordan's theory*.

3.2 fact. In Peano-Jordan's theory, a set is measurable if it can be well approximated by elementary sets from outside and from inside. More precisely, let $A \subset \mathbb{R}^n$ be a bounded set. We set

$$m_{*,PJ}(A) = \sup\{m(I) \ : \ I \text{ is an elementary set and } I \subset A\} \qquad \text{(inner measure)}$$
$$m^{*,PJ}(A) = \inf\{m(I) \ : \ I \text{ is an elementary set and } I \supset A\} \qquad \text{(outer measure)}$$

A set $A \subset \mathbb{R}^n$ is *Peano-Jordan measurable* if $m_{*,PJ}(A) = m^{*,PJ}(A)$. In this case we denote the Peano-Jordan measure of A as $m_{PJ}(A) = m_{*,PJ}(A) = m^{*,PJ}(A)$.

Peano-Jordan's theory works well with sets with *thin boundaries*, namely sets A with a boundary ∂A with $m^{*,PJ}(\partial A) = 0$. Moreover, it fits well with Riemann's integration theory, in a sense which is better explained as follows.

3.3 example. Let $f : [a,b] \to [0,+\infty)$ be Riemann integrable. Then, the subgraph of f

$$A = \{(x,y) \in \mathbb{R}^2 \;:\; a \leq x \leq b \,,\, 0 \leq y \leq f(x)\}$$

is Peano-Jordan measurable and

$$m_{PJ}(A) = \int_a^b f(x)dx.$$

Peano-Jordan's and Riemann's theories cover a fairly large class of sets and functions respectively. However, these theories lack in being well suited with respect to *σ-additivity* properties, i.e. they do not work well with countable unions of measurable sets. More precisely, consider a sequence of Peano-Jordan measurable sets $\{E_k\}_{k=1}^{+\infty}$, $E_k \subset \mathbb{R}^n$. In general, we are not guaranteed that the union $\bigcup_{k=1}^{+\infty}$ is Peano-Jordan measurable. This gap in the theory is only seemingly harmless. Its consequences on more advanced mathematical theories involving integration are indeed very serious. Moreover, Peano-Jordan theory cannot cover unbounded sets.

3.4 example. The set $A = [0,1] \cap \mathbb{Q}$ is countable. Write it as a sequences $A = \{x_k\}_k$ without repetitions. Clearly, each point has measure zero, therefore $m(\{x_k\}) = 0$. We would naturally expect that

$$m_{PJ}(A) = m_{PJ}\left(\bigcup_{k=1}^{+\infty}\{x_k\}\right) = \sum_{k=1}^{+\infty} m_{PJ}(\{x_k\}),$$

but this is *false*, since A is not even measurable according to Peano-Jordan theory. We leave the proof of this claim as an exercise.

3.5 exercise. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. Assume that $f_n \to f$ uniformly. Prove that

$$\lim_{n\to+\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

In 1902, measure theory was greatly improved by Henri Lebesgue, who formulated an extended version of Peano-Jordan's theory which fixed the above mentioned bugs.

3.2 An overview of Lebesgue measure theory

3.6 definition. Let $A \subset \mathbb{R}^n$ be a bounded open set. We set

$$m(A) = \sup\left\{m(I) \;:\; I \subset A \text{ and } I \text{ is an elementary set}\right\}.$$

Let $K \subset \mathbb{R}^n$ be a compact set. We set

$$m(K) = \inf\left\{m(I) \;:\; I \supset K \text{ and } I \text{ is an elementary set}\right\}.$$

For an arbitrary bounded subset $E \subset \mathbb{R}^n$ we define the *Lebesgue outer measure*

$$m^*(E) = \inf\{m(A) \;:\; A \supset E \text{ and } A \text{ is an open set}\},$$

and the *Lebesgue inner measure*

$$m_*(E) = \sup\{m(K) \;:\; K \subset E \text{ and } K \text{ is a compact set}\}.$$

A bounded subset $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable if $m^*(E) = m_*(E)$. If $E \subset \mathbb{R}^n$ is unbounded, we say that E is Lebesgue measurable if $E \cap B_R(0)$ is Lebesgue measurable for all $R \geq 0$, and set

$$m(E) = \lim_{R \to +\infty} m(E \cap B_R(0))$$

($m(E)$ may be infinite!).

It can be easily proven that this concept of measurability satisfies the so called *Boolean closure*, i. e. if $E, F \subset \mathbb{R}^n$ are measurable then so are $E \cup F$, $E \cap F$, $E \setminus F$. Moreover, the empty set is measurable with $m(\emptyset) = 0$, and $\mathbb{R}^n$ is measurable with $m(\mathbb{R}^n) = +\infty$.

3.7 exercise. Prove that a bounded set $E \subset \mathbb{R}^n$ is Lebesgue measurable if and only if for any $\epsilon > 0$ there exist $K_\epsilon \subset E \subset A_\epsilon$, with K_ϵ compact and A_ϵ open, such that $m(A_\epsilon \setminus K_\epsilon) < \epsilon$.

As we expect, Lebesgue measure *extends* Peano-Jordan measures.

3.8 exercise. Let $E \subset \mathbb{R}^n$ be Peano-Jordan measurable. Prove that E is Lebesgue measurable and that the two measures coincide. Hint: prove that

$$m^{*,PJ}(E) \geq m^*(E), \qquad m_{*,PJ}(E) \leq m_*(E).$$

In measure theory, sets with *zero measure* are particularly important. We say that a property holds *almost everywhere* in $\mathbb{R}^n$ if it holds outside a set with zero Lebesgue measure.

3.9 exercise. Prove that if $E \subset \mathbb{R}^n$ satisfies $m^*(E) = 0$ then E is Lebesgue measurable and $m(E) = 0$.

3.10 proposition. *If $E, F \subset \mathbb{R}^n$ are measurable, then*

(a) *$E \subset F$ implies $m(E) \leq m(F)$,*

(b) *$m(E \cup F) \leq m(E) + m(F)$,*

(c) *$m(E \cup F) = m(E) + m(F)$ if E and F are disjoint,*

(d) *$m(E \setminus F) = m(E) - m(F)$ if $E \supset F$ and $m(F) < +\infty$.*

Proof. Omitted. $\square$

As a consequence of (d) above, all bounded measurable sets have finite Lebesgue measure.

We mentioned above about gaps with σ-additivity properties in Peano-Jordan's theory. The next theorem collects fundamental properties of Lebesgue measure which marks a basic improvement in the theory.

3.11 theorem. *Let $\{E_k\}_k$ be a sequence of measurable sets in $\mathbb{R}^n$. Then, $\bigcap_{k=1}^{+\infty} E_k$ and $\bigcup_{k=1}^{+\infty} E_k$ are Lebesgue measurable. Moreover,*

(a) *$m\left(\bigcup_{k=1}^{+\infty} E_k\right) \leq \sum_{k=1}^{+\infty} m(E_k)$ (countable subadditivity),*

(b) *$m\left(\bigcup_{k=1}^{+\infty} E_k\right) = \sum_{k=1}^{+\infty} m(E_k)$ if the E_k's are pairwise disjoint (countable additivity),*

(c) *$m\left(\bigcup_{k=1}^{+\infty} E_k\right) = \lim_{k\to+\infty} m(E_k)$ if $E_k \subset E_{k+1}$ for all $k \in \mathbb{N}$ (continuity of Lebesgue measure w.r.t. increasing families of sets)*

(d) *$m\left(\bigcap_{k=1}^{+\infty} E_k\right) = \lim_{k\to+\infty} m(E_k)$ if $E_k \supset E_{k+1}$ for all $k \in \mathbb{N}$ and $m(E_1) < +\infty$ (continuity of Lebesgue measure w.r.t. decreasing families of sets).*

Proof. Omitted. $\square$

3.12 example. Let $E_k = [k, +\infty) \subset \mathbb{R}$. Clearly, $m([k, +\infty)) = +\infty$ for all k. We have

$$m\left(\bigcap_{k=1}^{+\infty} E_k\right) = m(\emptyset) = 0 \neq +\infty = \lim_{k\to+\infty} m(E_k),$$

so the assumption $m(E_1) < +\infty$ is needed in point (d) of the previous Theorem.

3.13 fact. Lebesgue measure is *translation invariant*. More precisely, given $E \subset \mathbb{R}^n$ a measurable set and given $h \in \mathbb{R}^n$, the set $E_h = \{y = x + h \ : \ x \in E\}$ is measurable and one has $m(E_h) = m(E)$.

3.14 example. In the example 3.4 we showed that the set $E = \mathbb{Q} \cap [0, 1]$ is not Peano-Jordan measurable. Due to Theorem 3.11 (b), E is actually Lebesgue measurable, since

$$E = \bigcup_{k=1}^{+\infty} \{x_k\}$$

where $\{x_k\}_{k=1}^{+\infty}$ is an enumeration (without repetitions) of the rational numbers in $[0,1]$. Therefore,

$$m(E) = \sum_{k=1}^{+\infty} m(\{x_k\}) = 0,$$

since all singletons $\{x_k\}$ have zero measure. Now, we left (b) Theorem 3.11 without a proof, but due to the importance of this example, let us prove that E is Lebesgue measurable and has zero measure. Let us fix $\epsilon > 0$. Let $\{x_k\}_{k=1}^{+\infty}$ the enumeration of E introduced above. We have

$$E \subset \bigcup_{k=1}^{+\infty} \left(x_k - \frac{\epsilon}{2^k}, x_k + \frac{\epsilon}{2^k} \right).$$

Now, we take for granted that

(a) Lebesgue outer measure is monotone, i.e. $E \subset F$ implies $m^*(E) \leq m^*(F)$ (easy exercise).

(b) Lebesgue outer measure is countably sub-additive, i.e. $m^* \left(\bigcup_{k=1}^{+\infty} E_k \right) \leq \sum_{k=1}^{+\infty} m(E_k)$ (exercise).

Hence, we obtain

$$m^*(E) \leq \sum_{k=1}^{+\infty} m \left(\left(x_k - \frac{\epsilon}{2^k}, x_k + \frac{\epsilon}{2^k} \right) \right) = 2\epsilon \sum_{k=1}^{+\infty} 2^{-k} = 2\epsilon,$$

and since $\epsilon > 0$ is arbitrary we obtain $m^*(E) = 0$, which proves the assertion.

Lebesgue's theory has the advantage of providing a class of measurable sets much larger than the one provided by Peano-Jordan's theory. A famous example due to Vitali shows, however, that there exists at least one subset of $\mathbb{R}$ which is not Lebesgue measurable.

In the remaining sections we shall use the expression *almost everywhere* for a property that holds everywhere but on a set of zero Lebesgue measure. We observe also that Lebesgue measure depends on the dimension d of the Euclidean space. Indeed, a segment with positive length has positive Lebesgue measure in dimension one and zero Lebesgue measure in dimension two. When necessary, we use the notation $m_d(A)$ for the measure of a set A to emphasize the dimension of the measure.

3.3 *Lebesgue integral*

Starting from the class of Lebesgue measurable sets, we introduce a class of functions on $\mathbb{R}^d$ which turns out to be the most suitable setting on which define a notion of integral. In what follows we adopt the notation $\overline{\mathbb{R}}$ to denote the extended real line $[-\infty, +\infty]$, given by $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

Let $\Omega \subset \mathbb{R}^n$, and let $f : \Omega \to \overline{\mathbb{R}}$. We say that f is measurable in Ω if

for all $\alpha \in \mathbb{R}$, the set $\{x \in \Omega \ : \ f(x) > \alpha\}$ is Lebesgue measurable.

Such a definition is equivalent to require, for all $\alpha \in \mathbb{R}$, one of the following:

- for all $\alpha \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \geq \alpha\}$ is Lebesgue measurable,
- for all $\alpha \in \mathbb{R}$, the set $\{x \in \Omega : f(x) \leq \alpha\}$ is Lebesgue measurable,
- for all $\alpha \in \mathbb{R}$, the set $\{x \in \Omega : f(x) < \alpha\}$ is Lebesgue measurable,
- for all $U \subset \mathbb{R}$ open, the set $\{x \in \Omega : f(x) \in U\}$ is Lebesgue measurable.

Clearly, continuous functions are measurable. Given f, g measurable and $c \in \mathbb{R}$, one has that the functions $f+g$, cf, fg, f/g (with $g \neq 0$), $\max\{f,g\}$, $\min\{f,g\}$, f^+, f^-, $|f|$ are measurable. Moreover, given a sequence $\{f_k\}_k$ of measurable functions, $\sup_{k\geq 1} f_k$, $\inf_{k\geq 1} f_k$, $\liminf_{k\to+\infty} f_k$, $\limsup_{k\to+\infty} f_k$, and $\lim_{k\to+\infty} f_k$ are measurable functions.

3.15 definition. A *simple function* $\phi : \mathbb{R}^n \to \mathbb{R}$ is a measurable function on $\mathbb{R}^n$ which attains only a finite number of values. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ be those values, and let $E_j = \{x \in \mathbb{R}^n : \phi(x) = \alpha_j\}$ for $j = 1, \ldots, k$. Then, ϕ can be represented as

$$\phi(x) = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{E_j}(x), \tag{32}$$

where

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases} \qquad \text{for a measurable set } A.$$

We observe that the above representation (32) is unique if we assume $E_i \cap E_j = \emptyset$ for $i \neq j$ and the constants α_i without repetitions. $\mathbf{1}_A$ above is called *indicator function*. The class of simple functions is closed under trivial operations such as sum, difference, multiplication by a real number.

We now define the notion of integral for simple functions. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a simple function which is bounded and zero outside a compact set of $\mathbb{R}^d$. Assume

$$\phi(x) = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{E_k}(x).$$

Then, we set

$$\int \phi(x)dx = \sum_{j=1}^{k} \alpha_j m(E_j).$$

The above definition is well posed, in the sense that it is independent of the choice of the representation of ϕ as finite combination of indicator functions. The integral of simple functions satisfies

$$\int (\phi + \lambda\psi)dx = \int \phi dx + \lambda \int \psi dx,$$

for all ϕ, ψ simple and fort all $\lambda \in \mathbb{R}$.

3.16 exercise. Let ϕ, ψ be simple functions. Assume $\phi \le \psi$ almost everywhere on $\mathbb{R}^d$. Prove that $\int \phi \le \int \psi$.

In Riemann's theory, the class of integrable functions is determined by the property of being well approximated by piecewise constant from above and from below. This requirement in general is quite selective, as it takes some nontrivial functions (such as the Dirichlet function) out of the set of integrable functions. In Lebesgue's theory, the minimal requirement of the function being measurable is essentially enough in order to compute the integral.

3.17 exercise. Let $f : \mathbb{R}^d \to [0, +\infty)$ be measurable, bounded, and zero outside the set $B_M(0)$. Then, prove that there exist two sequences ψ_k, ϕ_k of simple functions such that $\psi_k(x) = \phi_k(x) = 0$ for $x \notin B_M(0)$, $\psi_k \le f \le \phi_k$ for all k, and such that $\phi_k - \psi_k \to 0$ uniformly in $\mathbb{R}^d$. Hint: let $M > 0$ such that $\sup_{x \in \mathbb{R}^d} f(x) \le M < +\infty$ and $f(x) = 0$ for all $|x| \ge M$. For all $n \in \mathbb{N}$ and $k = 0, \ldots, 2^n$ consider the sets

$$E_k = \left\{ x \in \mathbb{R}^d \,:\, M\frac{k-1}{2^n} \le f(x) < M\frac{k}{2^n} \right\} \cap B_M(0),$$

and set $\psi_n(x) = \sum_{k=1}^{2^n} M\frac{k-1}{2^n}\mathbf{1}_{E_k}$ and $\phi_n(x) = \sum_{k=1}^{2^n} M\frac{k}{2^n}\mathbf{1}_{E_k}$. Prove that ϕ_n and ψ_n satisfy the assertion.

As a consequence of the above exercise, given $f : \mathbb{R}^d \to [0, +\infty)$ be measurable, bounded, and zero outside the set $B_M(0)$, as a consequence of exercise (3.16), one has

$$\int \phi_k dx - \int \phi_k dx = \int (\phi_k - \psi_k) dx \le \int \|\phi_k - \psi_k\|_\infty dx \le m(B_M(0))\|\phi_k - \psi_k\|_\infty,$$

and the right hand side goes to zero as $k \to +\infty$.

The above argument shows that, at least for measurable, nonnegative, bounded functions which are zero outside a compact set, approximation of the integral via simple functions works well both from outside and from below. Thus, we can define a notion of integral in this class in either directions. We choose to approximate from below.

3.18 definition (Lebesgue integral of a measurable function). Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. We set

$$\int f(x)dx = \sup\left\{ \int \phi(x)dx \,:\, \phi \text{ is simple and } \phi \le f \right\}.$$

Let $f : \mathbb{R}^d \to [-\infty, +\infty]$ be measurable and assume that at least one between $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ have finite integral. Then we set

$$\int f(x)dx = \int f_+(x)dx - \int f_-(x)dx.$$

A measurable function is called *summable* (or L^1) if $\int |f(x)|dx < +\infty$. For a given measurable set $E \subset \mathbb{R}^d$, we set

$$\int_E f(x)dx = \int f(x)\mathbf{1}_E(x)dx.$$

An elementary property which can be proven easily is *monotonicity* of the Lebesgue integral, that is, if $f \le g$ almost everywhere and $\int f dx$ and $\int g dx$ make sense, we have

$$\int f dx \le \int g dx \,.$$

The proof easily follows from the definition of Lebesgue integral and is left as an exercise.

Before proving more elementary properties of the Lebesgue integral, we need to prove a first result solving the limit-integral interchange property.

3.19 theorem (Beppo-Levi or monotone convergence). *Let $f_k : \mathbb{R}^d \to [0, +\infty]$ be a sequence of measurable functions. Assume that*

$$f_n(x) \le f_{n+1}(x) \qquad \text{almost everywhere.}$$

Then, there exists $f : \mathbb{R}^d \to [0, +\infty]$ measurable such that $f_n \to f$ almost everywhere, and

$$\lim_{n\to+\infty} \int f_n(x)dx = \int f(x)dx.$$

Proof. Let $A \subset \mathbb{R}^d$ be such that $f_n(x) \le f_{n+1}(x)$ for all $x \in \mathbb{R}^d \setminus A$ and $m(A) = 0$. We set $\widetilde{f_n}(x)$ to be equal to $f_n(x)$ on $\mathbb{R}^d \setminus A$ and 0 on A, so that $\widetilde{f_n}$ is monotone everywhere. We set

$$f(x) = \sup_{n\in\mathbb{N}} \widetilde{f_n}(x) \,.$$

Since $\widetilde{f_n}$ is measurable (exercise), then we get that f is also measurable. We observe that due to the monotonicity we have that $\lim_{n\to+\infty} \int f_n dx = \lim_{n\to+\infty} \int \widetilde{f_n} dx$ exists for sure. Moreover, since $\widetilde{f_n} \le f$ for all n, we immediately get

$$\lim_{n\to+\infty} \int f_n dx = \sup_{n\in\mathbb{N}} \int f_n dx \le \int f dx \,.$$

To prove the opposite inequality, let φ be a simple function which is zero outside a ball with $\varphi \le f$, without restriction $\varphi \ge 0$. For a $t \in (0, 1)$ we define the (Lebesgue measurable) set

$$E_n = \left\{ x \in \mathbb{R}^d \,:\, t\varphi(x) \le \widetilde{f}_(x) \right\} \,.$$

We represent the simple function φ as

$$\varphi(x) = \sum_{i=1}^{N} \alpha_i \mathbf{1}_{B_i} ,$$

for pairwise disjoint B_i's and distinct α_i's. We claim that $\bigcup_{i=1}^{N} E_n = \mathbb{R}^d$. To see this, let $x \in \mathbb{R}^d$. Since $\varphi(x) \leq f(x)$, we have $t\varphi(x) < f(x)$. Since $\widetilde{f}_n(x) \to f(x)$, for some $n \in \mathbb{N}$ we have $t\varphi(x) \leq \widetilde{f}_n(x)$, that is $x \in E_n$. Now, for a fixed n,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{f}_n(x)dx &\geq \int_{E_n} \widetilde{f}_n(x)dx \geq t \int_{E_n} \varphi(x)dx \\ &= t \int_{E_n} \sum_{i=1}^{N} \alpha_i \mathbf{1}_{B_i} = t \sum_{i=1}^{N} \alpha_i m(B_i \cap E_n) .\end{aligned}$$

Hence,

$$\sup_n \int_{\mathbb{R}^d} \widetilde{f}_n(x)dx \geq t \sum_{i=1}^{N} \alpha_i m(B_i \cap E_n) .$$

Since the family $B_j \cap E_n$ is increasing with respect to n, by continuity of the Lebesgue measure we can sent $n \to +\infty$ on the right hand side and get

$$\sup_n \int_{\mathbb{R}^d} \widetilde{f}_n(x)dx \geq t \sum_{i=1}^{N} \alpha_i m(B_i) = t \int_{\mathbb{R}^d} \varphi dx .$$

Due to the arbitrariness of t, we get

$$\lim_{n\to+\infty} f_n(x)dx = \int_{\mathbb{R}^d} \varphi dx$$

for all simple functions $\varphi \leq f$. By taking the sup with respect to φ we obtain the desired inequality. $\square$

We now prove more elementaty properties.

3.20 proposition. *Let f, g be measurable functions for which Lebesgue integral makes sense.*[7] *Let $\lambda, \mu \in \mathbb{R}$. Then,*

- $\int(\lambda f(x) + \mu g(x))dx = \lambda \int f(x)dx + \mu \int g(x)dx$.
- $\left|\int f(x)dx\right| \leq \int |f(x)|dx$.
- *If $f \geq 0$, $E \subset F$ and E, F are measurable, then* $\int_E f(x)dx \leq \int_F f(x)dx$.

[7] According to the above definition, this means either f is nonnegative or one between $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ have finite integral.

- *If $f \geq 0$ almost everywhere then*

$$m(\{x \in \mathbb{R}^d \,:\, f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int f(x)dx.$$

The proof is omitted and left as an exercise. Notice that Beppo-Levi is needed in the linearity property.

As expected, Lebesgue integral extends Riemann integral, i.e. if f is Riemann integrable then it is Lebesgue measurable and the two integrals coincide. This can be seen as a simple exercise by noticing for example that simple functions contain piecewise constant functions as a subset. On the other hand, there are functions which are Lebesgue integrable but not Riemann integrable, for example

$$f(x) = \mathbb{1}_{\mathbb{Q}\cap[0,1]},$$

the details are left as an exercise.

We now state some fundamental theorems regarding the Lebesgue integral.

3.21 theorem (Fubini). *Let $f(x,y)$ be a summable function on $\mathbb{R}^n \times \mathbb{R}^m$. Then,*

(i) The function $\mathbb{R}^n \ni x \mapsto f(x,y)$ is summable on $\mathbb{R}^n$ for almost all $y \in \mathbb{R}^m$.

(ii) The function $\mathbb{R}^m \ni y \mapsto \int f(x,y)dx$ is summable on $\mathbb{R}^m$ and we have

$$\int\limits_{\mathbb{R}^{n+m}} f(x,y)dxdy = \int dy \int f(x,y)dx. \tag{33}$$

If f is nonnegative and not necessarily summable, the same conclusion of (33) *holds if one of the three integrals*

$$\int\limits_{\mathbb{R}^{n+m}} f(x,y)dxdy, \qquad \int dy \int f(x,y)dx, \qquad \int dx \int f(x,y)dy$$

is finite.

The proof is omitted.

3.22 exercise. Let $f : \mathbb{R}^d \to [-\infty, +\infty]$ be a measurable function. Assume

$$\int\limits_E f(x)dx = 0$$

for all measurable sets $E \subset \mathbb{R}^d$. Prove that $f = 0$ almost everywhere.

The property proven for the Riemann integral in exercise 3.5 is a very important one. It is called *limit-integral exchange property*. A downside of Riemann integration is that such a property only holds in general under the quite strict assumption that the sequence f_n converges uniformly. The most natural class in which we would like to investigate such a property for Lebesgue integration is the class of function sequences $f_n : \mathbb{R}^d \to \mathbb{R}$ which are measurable and *converge almost everywhere* to some $f : \mathbb{R}^d \to \mathbb{R}$, i.e. such that $f_n(x) \to f(x)$ as $n \to +\infty$ for all $x \in \mathbb{R}^d \setminus A$ with $m(A) = 0$.

A first case in which the property is valid is when the sequence is monotone increasing almost everywhere, as proven in Beppo-Levi's theorem.nIn general, when the monotonicity property is not required for f_n the sequence does not necessarily converge almost everywhere, and the above limit exchange property is not necessarily true. However, under the assumption that the sequence is nonnegative, the following property can be proven.

3.23 theorem (Fatou's lemma). *Let $f_k : \mathbb{R}^d \to [0, +\infty]$ be a sequence of measurable functions. Then*

$$\int \left(\liminf_{n\to+\infty} f_n(x) \right) dx \leq \liminf_{n\to+\infty} \int f_n(x) dx.$$

Proof. We set $g_n(x) = \inf_{k \geq n} f_k(x)$ for all $n \in \mathbb{N}$. Clearly, we have

$$g_n(x) \leq g_{n+1}(x) \qquad \text{for all } n \in \mathbb{N} \text{ and for all } x \in \mathbb{R}^d.$$

Therefore, we can apply Beppo-Levi's theorem and get

$$\lim_{n\to+\infty} \int g_n(x) dx = \int \left(\lim_{n\to+\infty} g_n(x) \right) dx.$$

Since $g_n(x) \leq f_n(x)$ for all $n \in \mathbb{N}$, we get

$$\int \left(\lim_{n\to+\infty} g_n(x) \right) dx \leq \liminf_{n\to+\infty} \int f_n(x) dx.$$

The integrand in the left hand side above is $\liminf_{n\to+\infty} f_n(x)$. □

The following examples show that the strict inequality in Fatou's lemma occurs very often.

3.24 example (Concentration). Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = n\mathbf{1}_{[0,1/n]}(x).$$

The sequence f_n converges almost everywhere to $f = 0$. Moreover, it is easily seen that

$$\int_{\mathbb{R}} f_n(x) = 1 \qquad \text{for all } n \in \mathbb{N}.$$

Hence

$$0 = \int 0 dx < \liminf_{n\to+\infty} \int_{\mathbb{R}} f_n(x) dx = 1.$$

3.25 EXAMPLE (Travelling wave). Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \mathbf{1}_{[n,n+1]}(x).$$

The sequence f_n converges almost everywhere to $f = 0$. Moreover, it is easily seen that

$$\int_{\mathbb{R}} f_n(x) = 1 \qquad \text{for all } n \in \mathbb{N}.$$

Hence

$$0 = \int 0dx < \liminf_{n\to+\infty} \int_{\mathbb{R}} f_n(x)dx = 1.$$

The following theorem provides a quite general sufficient condition which removes the possibility that mass can be concentrated to one point or escape at infinity as it does in the previous examples.

3.26 THEOREM (Lebesgue's dominated convergence). *Let $f_k : \mathbb{R}^d \to [-\infty, +\infty]$ be a sequence of measurable functions. Suppose that*

(i) There exists a measurable function $f : \mathbb{R}^d \to [-\infty, +\infty]$ such that $f_n \to f$ almost everywhere.

(ii) There exists a summable function $g : \mathbb{R}^d \to [-\infty, +\infty]$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and for almost every $x \in \mathbb{R}^d$.

Then,

$$\int f(x)dx = \lim_{\to+\infty} \int f_n(x)dx.$$

Proof. For all $n \in \mathbb{N}$, let

$$h_n(x) = g(x) - f_n(x).$$

Since $h_n \geq 0$ almost everywhere, we can apply Fatou's lemma and get

$$\begin{aligned}\int \left(\liminf_{n\to+\infty} g(x) - f_n(x)\right) dx &= \int \left(\liminf_{n\to+\infty} h_n(x)\right) dx \\ \leq \liminf_{n\to+\infty} \int h_n(x)dx &= \liminf_{n\to+\infty} \int (g(x) - f_n(x))dx,\end{aligned}$$

and since $\int g(x)dx < +\infty$ we can use trivial properties of lim inf and lim sup and get

$$\int f(x)dx \geq \limsup_{n\to+\infty} \int f_n(x)dx. \tag{34}$$

We now set

$$H_n(x) = g(x) + f_n(x),$$

and since $H_n \geq 0$ almost everywhere we get by Fatou's lemma

$$\int \left(\liminf_{n\to+\infty} g(x) + f_n(x) \right) dx = \int \left(\liminf_{n\to+\infty} H_n(x) \right) dx$$
$$\leq \liminf_{n\to+\infty} \int H_n(x)dx = \liminf_{n\to+\infty} \int (g(x) + f_n(x))dx,$$

which implies

$$\int f(x)dx \leq \liminf_{n\to+\infty} \int f_n(x)dx. \tag{35}$$

(34) and (35) imply

$$\limsup_{n\to+\infty} \int f_n(x)dx \leq \int f(x)dx \leq \liminf_{n\to+\infty} \int f_n(x)dx,$$

and the assertion is proven since

$$\liminf_{n\to+\infty} \int f_n(x)dx \leq \limsup_{n\to+\infty} \int f_n(x)dx.$$

□

3.4 L^p *spaces*

In this subsection we introduce one of the main classes of Banach spaces used in functional analysis, i. e. the L^p *spaces*. They are constructed as function spaces on $\mathbb{R}^d$, and their theory makes use of the Lebesgue measure-integration theory developed above.

The theory we develop in this chapter will be defined for functions on $\mathbb{R}^d$ with values on $\mathbb{R}$, but everything can be easily generalised to the case of functions with values on $\mathbb{C}$.

3.27 DEFINITION. Let $p \in [1, +\infty)$, and let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. For a measurable function $f : \Omega \to \mathbb{R}$ we define the L^p norm of f on Ω as the (finite or infinite) number

$$\|f\|_{L^p(\Omega)} \doteq \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

Moreover, we set

$$C \doteq \{\alpha \in \mathbb{R} : \ |f(x)| \leq \alpha \text{ almost everywhere on } \Omega\}.$$

The L^∞ norm of f (also called the *essential supremum* of f) is defined as

$$\|f\|_{L^\infty(\Omega)} = \inf C.$$

The essential supremum of $|f|$ is the minimum *essential upper bound* for $|f|$, namely the minimum α such that $|f| \leq \alpha$ almost everywhere. It is easily seen that, in general

$$\|f\|_{L^\infty} \leq \sup_{x \in E} |f(x)|,$$

and very simple examples can be constructed in which the strict inequality holds above (basically we get a strict inequality anytime the f achieve its supremum $\overline{f}$ on a set of measure zero, and it is bounded above by a value strictly less than $\overline{f}$ elsewhere).

3.28 exercise. Prove that if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous then $\|f\|_{L^\infty(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)|$.

3.29 remark. A simple consequence of the above definition is that

$$|f(x)| \leq \|f\|_{L^\infty(E)} \qquad \text{almost everywhere on } E.$$

To see this, for all $k \in \mathbb{N}$ let

$$C_k = \{x \in E : \ |f(x)| \leq \|f\|_{L^\infty} + \frac{1}{k}\}.$$

Clearly, $m(E \setminus C_k) = 0$ for all $k \in \mathbb{N}$, because, for all $k \in \mathbb{N}$, the value $\|f\|_{L^\infty} + \frac{1}{k}$ is an upper bound for $|f|$ almost everywhere. Hence, $m\left(\bigcup_{k \in \mathbb{N}} (E \setminus C_k)\right) = 0$, and

$$\bigcup_{k \in \mathbb{N}} (E \setminus C_k) \supset E \setminus \{x \in E : \ |f(x)| \leq \|f\|_{L^\infty}\},$$

which implies

$$m\left(E \setminus \{x \in E : \ |f(x)| \leq \|f\|_{L^\infty}\}\right) = 0,$$

and therefore the assertion is proven.

Clearly, we would like to define a norm by means of the L^p and L^∞ norms. The problem is that, in general, $\|f\|_{L^p} = 0$ does not imply $f \equiv 0$, which is one of the axioms of a norm. Indeed, for all $p \geq 1$ the statement $\|f\|_{L^p(E)} = 0$ only implies that $|f(x)| \neq 0$ almost everywhere on E, but f could still be nonzero on a set of null measure. We will see in a few pages how to bypass this problem.

Let $p \in [1, +\infty]$. We define the *conjugate* of p is the number p' defined by

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $1/+\infty = 0$. In particular, 1 is the conjugate of $+\infty$ and vice versa.

3.30 exercise (Young's inequality). Let $p \in [1, +\infty)$ and let p' be its conjugate.

Let $a, b \geq 0$ be two positive numbers. Then,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Solution. If $ab = 0$ there is nothing to prove. Assume $a > b > 0$. Set $A = a^p$ and $B = b^{p'}$. We need to prove that

$$A^{1/p} B^{1/p'} \leq \frac{A}{p} + \frac{B}{p'}.$$

Multiplication by $1/B$ makes the above inequality equivalent to

$$\frac{1}{p}\left(\frac{A}{B}\right) + \frac{1}{p'} \geq \left(\frac{A}{B}\right)^{1/p}$$

where we have used $\frac{1}{p'} - 1 = -\frac{1}{p}$. Now, set $t = \frac{A}{B} \geq 1$. The above becomes equivalent to proving that

$$\frac{1}{p} t + \frac{1}{p'} \geq t^{1/p} \qquad \text{for all } t \geq 1.$$

But the function $\phi(t) = \frac{1}{p}t + \frac{1}{p'} - t^{1/p}$ satisfies $\phi(1) = 0$, $\phi'(t) = \frac{1}{p} - \frac{1}{p} t^{\frac{1}{p}-1}$, which is ≥ 0 for $t \geq 1$. Therefore, $\phi(t) \geq 0$ for all $t \geq 1$, which proves the assertion.

3.31 THEOREM (Hölder inequality). *Let $f, g : E \to \mathbb{R}$ be measurable functions, and let $p, q \in [1, +\infty]$ be conjugate. Then,*

$$\|fg\|_{L^1(E)} \leq \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

Proof. If $p = 1$ and $q = +\infty$, we have

$$\|fg\|_{L^1(E)} = \int_E |f(x)g(x)| dx \leq \|g\|_{L^\infty} \int |f(x)| dx = \|g\|_{L^\infty} \|f\|_{L^1},$$

where the first inequality is justified by the fact that g can be redefined on a set of measure zero in a way that $g \leq \|g\|_{L^\infty}$ everywhere, and this does not affect the integral.

In the general case $p > 1$, the statement is trivial if either f or g are zero almost everywhere. Otherwise, we clearly have $\|f\|_{L^p} > 0$ and $\|g\|_{L^q} > 0$. For a fixed $\alpha > 0$ we have

$$|f(x)g(x)| = \left|\frac{f(x)}{\alpha}\right| |\alpha g(x)| \leq \frac{1}{p}\left|\frac{f(x)}{\alpha}\right|^p + \frac{1}{q} |\alpha g(x)|^q,$$

where we have used Young's inequality (Exercise 3.70). By integrating the above inequality on E we get

$$\|fg\|_{L^1(E)} \le \frac{1}{p}\frac{1}{\alpha^p}\|f\|^p_{L^p(E)} + \frac{1}{q}\alpha^q\|g\|^q_{L^q(E)}.$$

We now choose α such that the two terms on the above right hand side are equal, namely

$$\alpha := \frac{\|f\|_{L^p}^{\frac{1}{q}}}{\|g\|_{L^q}^{\frac{1}{p}}},$$

which yields

$$\|fg\|_{L^1(E)} \le \frac{1}{p}\frac{\|g\|_{L^q}}{\|f\|_{L^p}^{p/q}}\|f\|^p_{L^p} + \frac{1}{q}\frac{\|f\|_{L^p}}{\|g\|_{L^q}^{q/p}}\|g\|^q_{L^q},$$

and the definition of p and q implies the last term above equals $\|f\|_{L^p}\|g\|_{L^q}$. □

3.32 remark. Hölder inequality can be also rephrased as follows:

$$\int_E |f(x)|^\alpha |g(x)|^\beta dx \le \left(\int_E |f(x)|dx\right)^\alpha \left(\int_E |g(x)|dx\right)^\beta,$$

provided $\alpha + \beta = 1$.

3.33 exercise (Discrete Hölder inequality). Let

$$x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n,$$

and let $p, q \in [1, +\infty)$ with $1/p + 1/q = 1$. Prove that

$$\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}.$$

3.34 theorem (Minkowski's inequality). *Let $f, g : E \to \mathbb{R}$ be measurable functions. Let $p \in [1, +\infty]$. Then,*

$$\|f+g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)}.$$

Proof. The case $p = +\infty$ is trivial, since for all $x \in E$ one has

$$|f(x) + g(x)| \le |f(x)| + |g(x)|,$$

and the right hand side is controlled by $\|f\|_{L^\infty} + \|g\|_{L^\infty}$ almost everywhere on E. This implies the assertion.

The case $p = 1$ is straightforward.

Let $p \in (1, +\infty)$. The statement is trivially satisfied if either $\|f\|_{L^p}$ or $\|g\|_{L^p}$

equals $+\infty$. Therefore, assume the are both finite. We observe that

$$
\begin{aligned}
|f(x)+g(x)|^p &\le |f(x)+g(x)|^{p-1}(|f(x)|+|g(x)|) && (36)\\
&\le |f(x)+g(x)|^{p-1}|f(x)| + |f(x)+g(x)|^{p-1}|g(x)|. && (37)
\end{aligned}
$$

Integrating on E yields, via the Hölder inequality,

$$
\begin{aligned}
&\int_E |f(x)+g(x)|^{p-1}|f(x)|dx\\
&\le \left(\int_E |f(x)+g(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}} = \|f+g\|_{L^p}^{p-1}\|f\|_{L^p}.
\end{aligned}
$$

By performing the same manipulation on the last term in (36), we get

$$
\int_E |f(x)+g(x)|^p dx = \|f+g\|_{L^p}^p \le \|f+g\|_{L^p}^{p-1}\left(\|f\|_{L^p}+\|g\|_{L^p}\right),
$$

which proves the assertion. □

Now, clearly the L^p norm verifies

- $\|f\|_{L^p} \ge 0$,
- $\|\lambda f\|_{L^p} = |\lambda|\|f\|_{L^p}$,
- $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$,

in the class of measurable functions on which the above quantities are finite. But this is not enough to make $\|\cdot\|_{L^p}$ a norm on such space, because in general $\|f\|_{L^p} = 0$ does not imply $f \equiv 0$. Please notice that the underlying linear space on which we are defining (or trying to define) a norm here is the space of measurable functions on $\mathbb{R}^d$ such that $\|f\|_{L^p} < +\infty$ (verify as an exercise that such a set is a linear space).

3.35 FACT. Let us recall a simple fact from set theory. Given a set X, an *equivalence* on X is a subset E of $X \times X$ such that

(i) $(x,x) \in E$ for all $x \in E$,

(ii) $(x,y) \in E$ if and only if $(y,x) \in E$,

(iii) If (x,y) and (y,z) are in E, then $(x,z) \in E$.

We use the notation $x \sim y$ to denote $(x,y) \in E$. Given an equivalence on X, and given $x \in X$, we set

$$
[x] = \{y \in X : x \sim y\},
$$

called the *equivalence class* of x. We call $X/\sim$ the set of all equivalence classes for the relation E. Such set is called the *quotient set*.

3.36 exercise. Let X be a real (or complex) linear space. Let $\sim$ be an equivalence on X. Given $[x], [y] \in X/\sim$ and $\lambda, \mu \in \mathbb{R}$, set

$$\lambda[x] + \mu[y] = [\lambda x + \mu y] \in X/\sim .$$

Prove that such a definition is well posed (i.e. the class $[x] + [y]$ does not depend on the choice of the vectors x, y. Prove that $X/\sim$ is a real (or complex) linear space with the above defined operation.

3.37 definition. Let $E \subset \mathbb{R}^d$ be a measurable set and $p \in [1, +\infty]$. We call $\mathcal{L}^p(E)$ the set of measurable functions $f : E \to \mathbb{R}$ such that $\|f\|_{L^p} < +\infty$. Now, we set the following equivalence on $\mathcal{L}^p(E)$. Let $f, g \in \mathcal{L}^p(E)$. We say that $f \sim g$ if $f(x) = g(x)$ for almost every $x \in E$.[8] We set

$$L^p(E) = \mathcal{L}^p(E)/\sim,$$

i. e. $L^p(E)$ is the quotient (vector) space of $\mathcal{L}^p(E)$ through the relation $\sim$[9]. For a given equivalence class $[f] \in L^p(E)$, we define the norm of $[f]$ as $\|f\|_{L^p(E)}$ for an arbitrary representant $f \in [f]$. From now on, by abuse of notation, we shall confuse $[f]$ and its representant f. The space $L^p(E)$ is the *L^p space on E*.

3.38 exercise. Prove that the norm of $f \in L^p(E)$ is well defined.

3.39 remark. Clearly, the L^p norm on $L^p(E)$ is now a norm, since all the 'good' properties proven above are easily inherited by the norm on the quotient space, and furthermore one has $\|f\|_{L^p} = 0$ implies $f = 0$ almost everywhere, hence $[f] = 0$. Therefore, $L^p(E)$ is a normed space.

We shall say that a sequence $f_n \in L^p(E)$ converges in L^p to $f \in L^p$ if $\|f_n - f\|_{L^p} \to 0$ as $n \to +\infty$.

We now prove that L^p spaces are *complete*, i.e. they are Banach spaces.

3.40 theorem (Riesz-Fisher theorem). *Let $E \subset \mathbb{R}^d$ be a measurable set and $p \in [1, +\infty]$. Then the space $L^p(E)$ is Banach space. Moreover, if $p \in [1, +\infty)$ and $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(E)$, then there exist two functions $f, h \in L^p(E)$ and a subsequence f_{n_k} of f_n such that*

(a) $|f_{n_k}(x)| \le h(x)$ almost everywhere on E,

(b) $f_{n_k} \to f$ almost everywhere on E.

Proof. Postponed.

The theorem above is quite important. Apart from stating that L^p spaces are complete, it investigates the interplay between L^p convergence and almost everywhere convergence. More precisely, it says that if a sequence f_n converges in L^p to some f, then f_n has a subsequence that converges almost

[8] Prove that $\sim$ is actually an equivalence!

[9] All the vector space operations on $\mathcal{L}^p(E)$ are inherited by the quotient space by considering operations between representant. For instance, given $[f], [g] \in L^p$, $[f] + [g]$ is the equivalence class of $[f + g]$. Prove that such operation is well defined. Define similarly $\lambda[f]$ for some $\lambda \in \mathbb{R}$.

everywhere. The next example shows that, in general, convergence in L^p does not imply convergence almost everywhere of the *whole* sequence.

3.41 EXAMPLE. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = 0$, $f_1(x) = \mathbf{1}_{[0,1/2)}(x)$, $f_2(x) = \mathbf{1}_{[1/2,1]}(x)$, and for general $k \geq 1$, $k \in \mathbb{N}$, and for all $n = 2^{k-1} + 1, \dots, 2^k$,

$$f_n(x) = \mathbf{1}_{[n2^{-k}-1, n2^{-k})}(x).$$

One can easily see that the $L^1([0,1])$ norm of f_n tends to zero as $n \to +\infty$, so that $f_n \to 0$ in $L^1([0,1])$. However, for every $x \in [0,1]$, the set of integers n such that $f_n(x) = 1$ is infinite, and therefore $f(x)$ cannot converge to zero. This is true for *every* $x \in [0,1]$. Therefore, f_n converges to zero on the empty set, so it does not converge to zero on a set of measure 1. Hence, it is not true that f_n converges to zero almost everywhere.

On the other hand, does almost everywhere convergence imply L^p convergence? This is also not true in general, as one can deduce from the example $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \mathbf{1}_{[n,n+1)}(x),$$

in which f_n converges to zero almost everywhere but not in L^1 (Exercise).

3.42 DEFINITION (Support of a continuous function). Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $f \in C(\Omega)$[10]. The *support of f in Ω* is the set

$$\operatorname{spt}(f) = \{x \in \Omega \, : \, f(x) \neq 0\}.$$

If $\operatorname{spt}(f)$ is compact, we say that f is *compactly supported*. The space of compactly supported functions on Ω is denoted by $C_c(\Omega)$. We notice that $C_c(\Omega) \subset C_b(\Omega) \subset C(\Omega)$.

We now recall the notion of distance between sets.

3.43 DEFINITION. Let $x \in \mathbb{R}^d$, $A, B \subset \mathbb{R}^d$. We set

$$d(x, A) = \inf\{\|x - y\| \, : \, y \in A\},$$

and

$$d(A, B) = \inf\{\|x - y\| \, : \, x \in A, y \in B\}.$$

Here $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$.

3.44 EXERCISE. The distance function defined above has the following properties.

- $d(A, B) = \inf_{x \in A} d(x, B) = \inf_{y \in B} d(x, A)$ (exercise).

[10] We recall that $C(\Omega)$ is the space of continuous functions from Ω to $\mathbb{R}$

- The map $\mathbb{R}^d \ni x \mapsto d(x,A)$ is continuous, indeed, the map is *Lipschitz continuous* with Lipschitz constant 1, i. e.

$$|d(x,A) - d(y,A)| \leq \|x - y\|.$$

Left as an exercise. Hint: take an arbitrary point $z \in A$ and use the triangular inequality.

- If $K, C \subset \mathbb{R}^d$ with K compact, C closed, and $K \cap C = \emptyset$, then $d(K,C) > 0$ (exercise).

- Let $A \subset \mathbb{R}^d$ and $\delta > 0$. Set

$$A_\delta = \{x \in \mathbb{R}^d \; : \; d(x,A) \leq \delta\}.$$

Prove that $A \subset A_\delta$ and A_δ is closed. Moreover, if $K \subset \mathbb{R}^d$ is compact then K_δ is compact. This is an easy exercise.

- As a consequence of the above exercises, let $\Omega \subset \mathbb{R}^d$ be open, and let $K \subset \Omega$ be compact. Let $\delta_0 = d(K, \mathbb{R}^d \setminus \Omega)$. Clearly, $\delta_0 > 0$. Hence, for all $\delta < \delta_0$ one has

$$K \subset \overset{\circ}{K}_\delta \subset K_\delta \subset \Omega.$$

The next proposition is a special case of a more general result in topology known as *Urysohn's lemma*[11].

3.45 proposition. *Let $K \subset \Omega \subset \mathbb{R}^d$, with K compact and Ω open. Then, there exists $\varphi \in C_c(\Omega)$ such that $\varphi(x) = 1$ for all $x \in K$ and $0 \leq \varphi(x) \leq 1$ for all $x \in \Omega$.*

Proof. Let δ such that $0 < \delta < d(K, \mathbb{R}^d \setminus \Omega)$. Set

$$\varphi(x) = \frac{d(x, \mathbb{R}^d \setminus K_\delta)}{d(x, \mathbb{R}^d \setminus K_\delta) + d(x,K)}.$$

Clearly, $d(x, \mathbb{R}^d \setminus K_\delta) + d(x,K) \neq 0$ for all $x \in \Omega$. Indeed, if $d(x,K) = 0$ then $x \in K$, and therefore $d(x,y) \geq \delta$ for all $y \in \mathbb{R}^d \setminus K_\delta$. Moreover, $\varphi(x) \in [0,1]$ for all $x \in \Omega$, and $x \in K$ implies $d(x,K) = 1$ and $\varphi(x) = 1$. Finally, $\varphi(x) \neq 0$ only if $d(x, \mathbb{R}^d \setminus K_\delta) \neq 0$, which is equivalent to $x \in K_\delta$. Hence, the support of φ is compact in Ω. □

In what follows we shall denote with $\mathcal{S}(\Omega)$ the space of simple functions on Ω which are zero outside a bounded set.

3.46 theorem (Density of C_c in L^p). *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

(i) The space $\mathcal{S}(\Omega)$ is dense in L^p if $p \in [1, +\infty)$.

(ii) $C_c(\Omega)$ is dense in $L^p(\Omega)$ if $p \in [1, +\infty)$.

[11] `http://en.wikipedia.org/wiki/Urysohn's_lemma`

(iii) $C_c(\Omega)$ *is not dense in* $L^\infty(\Omega)$. $\mathcal{S}(\Omega)$ *is dense in* $L^\infty(\Omega)$ *if* Ω *is bounded.*

Proof. Proof of (i). Let $f \in L^p(\Omega)$. We have to construct a sequence of simple functions $\phi_j \in \mathcal{S}(\Omega)$ with $\|\phi_j - f\|_{L^p} \to 0$ as $j \to +\infty$. Assume first that $f \geq 0$. We know from Exercise 3.17 that there exists a sequence of nonnegative simple functions $\phi_j \in \mathcal{S}(\Omega)$ with $\phi_j \nearrow f$ almost everywhere in case f is zero outside a bounded set. In the general case of $f \geq 0$, for a given $n \in \mathbb{N}$ there exists a sequence $\phi_{n,j} \nearrow f\mathbf{1}_{B_n(0)}$. Consider the diagonal sequence $\phi_{j,j} \in \mathcal{S}(\Omega)$. Let $\epsilon > 0$. For almost every $x \in \Omega$ one has $x \in B_j(0)$, and hence $|f(x) - \phi_{j,j}| \leq \epsilon$ for j large enough. Hence, the claim is true also for a general $f \geq 0$. Now, this implies

$$0 \leq |f - \phi_j|^p \leq |f|^p,$$

almost everywhere on Ω. Therefore, we can apply Lebesgue dominated convergence theorem 3.26 to get

$$\int |f(x) - \phi_j(x)|^p dx \to 0 \qquad \text{as } j \to +\infty.$$

The general case f sign changing can be solved by splitting $f = f_+ - f_-$, constructing sequences of simple functions as above for f_+ and f_-, and applying the previous step.

Proof of (ii). Let $f \in L^p(\Omega)$ and let $\epsilon > 0$. Due to (i) there exists a simple function ϕ on Ω such that $\|f - \phi\|_{L^p} \leq \frac{\epsilon}{2}$. The proof will be completed once we find a continuous function g on Ω such that $\|g - \phi\|_{L^p} \leq \frac{\epsilon}{2}$. Assume first that $\phi = \alpha\mathbf{1}_F$ for some measurable bounded set $F \subset \Omega$ and some $\alpha \in \mathbb{R}$. Fix $\sigma > 0$. Let $K \subset F \subset A$, K compact and A open, such that $m(A) - m(K) < \sigma$. From Proposition 3.45, we know that there exists a function $\tilde{g} \in C_c(A)$ such that $0 \leq \tilde{g} \leq 1$ and $\tilde{g} = 1$ on K. Let $g = \alpha\tilde{g}$. We have

$$\|g - \phi\|_{L^p}^p = \int_\Omega |g(x) - \phi(x)|^p dx = \int_\Omega |\alpha\tilde{g} - \alpha\mathbf{1}_F|^p dx \leq \alpha^p m(A \setminus K) \leq \alpha^p \sigma,$$

and choosing $\sigma = (\epsilon/2\alpha)^p$ one has $\|g - \phi\|_{L^p} \leq \frac{\epsilon}{2}$. Assume now that

$$\phi = \sum_{j=1}^N \alpha_j \mathbf{1}_{F_j},$$

with F_j measurable and bounded sets. From the previous case we can find continuous functions g_j such that

$$\|g_j - \mathbf{1}_{F_j}\|_{L^p} \leq \frac{\epsilon}{2\sum_{j=1}^N |\alpha_j|}.$$

Set $g = \sum_{j=1}^{N} \alpha_j g_j$, we have

$$\|g - \phi\|_{L^p} = \|\sum_{j=1}^{N} \alpha_j (g_j - \mathbf{1}_{F_j})\|_{L^p} \leq \sum_{j=1}^{N} \|\alpha_j|g_j - \mathbf{1}_{F_j}|\|_{L^p} \leq \frac{\epsilon}{2},$$

and the assertion (ii) is proven.

Proof of (iii). $C_c(\Omega)$ cannot be dense in $L^\infty(\Omega)$. Indeed, take $f \in L^\infty(\Omega)$ discontinuous at one point. The density property would imply that there exists a sequence of continuous functions f_j on Ω that converge in L^∞ to f. But for continuous functions the convergence in L^∞ is equivalent to the uniform convergence, and this is in contradiction with a well known convergence property of sequences of functions.

If Ω is bounded, then the statement that simple functions are dense in L^∞ is an immediate consequence of exercise 3.17. □

3.47 theorem (Separability of L^p). *$L^p(\Omega)$ is separable if $p \in [1, +\infty)$. $L^\infty(\Omega)$ is not separable.*

Proof. Case $p < +\infty$. For simplicity let us assume $\Omega = \mathbb{R}^d$ (the general case is a simple consequence of this special case). Let $\mathcal{R}$ denote the countable family of sets in $\mathbb{R}^d$ of the form $R = \Pi_{k=1}^{d}(a_k, b_k)$ with $a_k, b_k \in \mathbb{Q}$. Let $\mathcal{E}$ denote the vector space over $\mathbb{Q}$ generated by the functions $\{\mathbf{1}_R\}_{R\in\mathcal{R}}$, that is, $\mathcal{E}$ consists of all finite linear combinations with rational coefficients of indicator functions of sets in $\mathcal{R}$. It is easily seen that $\mathcal{E}$ is a countable set.

We claim that $\mathcal{E}$ is dense in $L^p(\mathbb{R}^d)$. Let $f \in L^p(\mathbb{R}^d)$ and let $\epsilon > 0$. From Theorem 3.46 there exists $f_1 \in C_c(\mathbb{R}^d)$ such that $\|f - f_1\|_{L^p} \leq \epsilon$. Let $R \in \mathcal{R}$ be any cube containing the support of f_1. It is easy to construct a function $f_2 \in \mathcal{E}$ such that $\|f_1 - f_2\|_{L^p} \leq \varepsilon$ and f_2 vanishes outside R. Indeed, it suffices to split R into small cubes of $\mathcal{R}$ where the oscillation of f_1 is less than an arbitrary $\delta > 0$ (f_1 is uniformly continuous on R!). Therefore we have

$$\|f_1 - f_2\|_{L^p} \leq \|f_1 - f_2\|_{L^\infty} m(R)^{1/p} < \delta m(R)^{1/p}.$$

We conclude that $\|f - f_2\|_{L^p} < 2\epsilon$ provided $\delta > 0$ is chosen so that $\delta m(R)^{1/p} < \epsilon$.

Case $p = +\infty$. If we prove that there exists a family of open balls in $L^\infty(\Omega)$ which are pairwise disjoint and with an uncountable cardinality, the proof will be completed. Indeed, if such property is satisfied, any dense subset S in L^∞ should have at least one element in each of the above open balls, and this makes it impossible for S to be countable. Now, given two open balls $B, B' \subset \Omega$, assuming that $B \neq B'$, one has

$$\|\mathbf{1}_B - \mathbf{1}_{B'}\|_{L^\infty} = 1,$$

and the proof is an easy exercise. Now, for a given ball $B \subset \Omega$, set

$$U_B = \{g \in L^\infty : \|g - \mathbf{1}_B\|_{L^\infty} < \frac{1}{2}\}.$$

Clearly, the family

$$\mathcal{U} = \{U_B \ : \ B \text{ is an open ball in } \Omega\}$$

is more than countable, and every two distinct elements in $\mathcal{U}$ are disjoint. This proves the assertion. $\square$

3.48 remark. Let $\Omega \subset \mathbb{R}^d$ be open, and let $p, q \in [1, +\infty]$ with $p \leq q$. Is there any relation between L^p and L^q? More presicely, is one of the two spaces a subset of the other one? In general the answer is negative. As an example, let $p = 1$ and $q = 2$, and let $\Omega = (0, +\infty) \subset \mathbb{R}$. Take $f(x) = \frac{1}{1+x}$. Clearly, $f \notin L^1(\Omega)$, where as $f \in L^2(\Omega)$. Now, let $g(x) = \frac{1}{\sqrt{x}}\mathbf{1}_{(0,1)}$. Clearly, $g \in L^1(\Omega)$ but $g \notin L^2(\Omega)$. Hence, $L^1(\Omega)$ is not a subset of $L^2(\Omega)$ and $L^2(\Omega)$ is not a subset of $L^1(\Omega)$.

On the other hand, if $m(\Omega) < +\infty$, the L^p spaces are ordered. Indeed, let $p \leq q$: then $L^q(\Omega) \subseteq L^p(\Omega)$. To see this, assume first $q < +\infty$. We compute

$$\int_\Omega |f|^p dx = \int_{|f|\geq 1} |f|^p dx + \int_{|f|<1} |f|^p dx \leq \int_{|f|\geq 1} |f|^q dx + m(\Omega) \leq \int |f|^q dx + m(\Omega),$$

and hence $\|f\|_{L^q} < +\infty$ implies $\|f\|_{L^q} < +\infty$. Now, let us consider the case $q = +\infty$. We have

$$\int_\Omega |f|^p \leq \|f\|^p_{L^\infty} m(\Omega),$$

and this proves the assertion.

Having defined the new family of functional spaces $L^p(\Omega)$ for $p \in [1, +\infty]$ allows to consider a new notion of convergence for sequences of functions. Given $\Omega \subset \mathbb{R}^d$ a measurable set, we have that $L^p(\Omega)$ is a complete normed space, i.e. a Banach space. As such, it encompasses a notion of convergence. A sequence $f_n \in L^p(\Omega)$ converges to f in L^p if $\|f_n - f\|_{L^p} \to 0$ as $n \to +\infty$. The above remark shows that there is no relationship between convergence in $L^p(\Omega)$ and convergence in $L^q(\Omega)$ for $p \neq q$ unless Ω has finite measure. In this case, the convergence in L^1 is weaker than any other L^p convergence, whereas the L^∞ one is the strongest. The uniform convergence is stronger than the L^∞ convergence on an arbitrary measurable set Ω (even if Ω is unbounded).

3.5 *Convolution, regularisation and L^p_{loc} spaces.*

We first define the convolution product of a function $f \in L^1(\mathbb{R}^d)$ with a function $g \in L^p(\mathbb{R}^d)$.

3.49 theorem (Young). *Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ with $p \in [1, +\infty]$. Then, for almost every $x \in \mathbb{R}^d$ the function $y \mapsto f(x-y)g(y)$ is summable on $\mathbb{R}^d$ and we* *define*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy\,.$$

In addition $f * g \in L^p(\mathbb{R}^d)$ *and we have*

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}\,.$$

Proof. The conclusion is trivial if $p = +\infty$. We now consider the case $p = 1$. Set $F(x,y) = f(x-y)g(y)$. For a.e. $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} |F(x,y)|dx = |g(y)| \int_{\mathbb{R}^d} |f(x-y)|dy = |g(y)| \|f\|_{L^1(\mathbb{R}^d)} < +\infty\,.$$

Moreover,

$$\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} |F(x,y)|dx = \|g\|_{L^1(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)} < +\infty\,.$$

From Fubini's theorem we get that $F \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ and that we can exchange the order of integration to obtain the desired assertion.

Let us now consider the case $p > 1$. From the previous case, for almost every $x \in \mathbb{R}^d$ the function

$$y \mapsto |f(x-y)||g(y)|^p$$

is summable, that is

$$y \mapsto |f(x-y)|^{1/p}|g(y)|$$

belongs to $L^p(\mathbb{R}^d)$. Hence, we apply Hoelder's inequality to obtain that the function

$$y \mapsto |f(x-y)||g(y)| = |f(x-y)|^{1/p'}|f(x-y)|^{1/p}|g(y)|$$

belongs to $L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |f(x-y)||g(y)|dy \leq \|f\|_{L^1(\mathbb{R}^d)}^{1/p'} \left(\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy \right)^{1/p},$$

which implies

$$|f * g(x)|^p \leq \|f\|_{L^1(\mathbb{R}^d)}^{p/p'} (|f| * |g|^p)(x)\,.$$

The first case then implies the left hand side is summable and

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|^{p/p'}_{L^1(\mathbb{R}^d)} \|f\|_{L^1(\mathbb{R}^d)} \|g\|^p_{L^p(\mathbb{R}^d)}$$

which implies the assertion. □

In the sequel we shall denote

$$\check{f}(x) = f(-x).$$

3.50 remark. Let $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, and $h \in L^{p'}(\mathbb{R}^d)$. Then we have

$$\int_{\mathbb{R}^d} (f * g) h dx = \int_{\mathbb{R}^d} g(\check{f} * h) dx\,.$$

To prove this, let

$$F(x,y) = f(x-y)g(y)h(x)$$

which belongs to $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ because

$$\int_{\mathbb{R}^d} |h(x)| \int_{\mathbb{R}^d} |f(x-y)||g(y)| dy < +\infty$$

in view of Hoelder's inequality and the previous Theorem. Moreover,

$$\int_{\mathbb{R}^d} (f * g)(x) h(x) dx = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} F(x,y) dy = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} F(x,y) dx = \int_{\mathbb{R}^d} g(y)(\check{f} * h)(y) dy\,.$$

We now want to refine our concept of support for L^p functions. The one we have defined so far only applies to continuous functions. The problem with L^p space is that this is a set of equivalence classes, therefore the usual definition does not apply. As an example, consider the indicator function of the set $D = [0,1] \cap \mathbb{Q}$. The support of this function with the usual definition would be $[0,1]$, because the latter is the closure of D. However, this function is almost everywhere equal to zero, and the support of zero is the empty set. Hence, this definition is not well posed. To bypass this problem, we proceed as follows.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be any function. Consider the family $\{\omega_i\}_{i \in I}$ of all open sets on which $f = 0$ almost everywhere. Set $\omega = \bigcup_{i \in I} \omega_i$. We claim that $f = 0$ almost everywhere on ω. To see this, consider a countable family of open sets O_n in $\mathbb{R}^d$ such that every open set can be written as union of some O_n. This is doable for instance by considering open balls with center having rational components and rational radius. Write $\omega_i = \bigcup_{n \in A_i} O_n$, which implies $\omega = \bigcup_{n \in B} O_n$ where $B = \bigcup_{i \in I} A_i$. For all $n \in B$ we have $n \in A_i$ for some $i \in I$. Since f is zero almost everywhere on ω_i, we have $f = 0$ almost everywhere on O_n. Then, f is zero almost everywhere on every O_n included in ω, which implies that $f = 0$ almost everywhere on ω.

We then set, by definition, $\mathrm{supp}(f)$ as $\mathbb{R}^d \setminus \omega$. We immediately see that if $f_1 = f_2$ almost everywhere then $\mathrm{supp}(f_1) = \mathrm{supp}(f_2)$. This is due to the fact that if f_1 and f_2 coincide almost everywhere they cannot differ on any open set.

3.51 exercise. Check that the above definition coincides with the usual one in case f is continuous.

3.52 proposition. *Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ with $1 \in [1, +\infty]$. Then,*

$$\mathrm{supp}(f * g) \subset \overline{\mathrm{supp}(f) + \mathrm{supp}(g)}.$$

Proof. Fix $x \in \mathbb{R}^d$ and consider

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{(x-\mathrm{supp}(f))\cap \mathrm{supp}(g)} f(x-y)g(y)dy\,.$$

Now, assume $x \notin \mathrm{supp}(f) + \mathrm{supp}(g)$. Then $x - \mathrm{supp}(f)$ and $\mathrm{supp}(g)$ have empty intersection, and therefore $(f * g)(x) = 0$. Thus, $f * g = 0$ on $\mathbb{R}^d \setminus (\in \mathrm{supp}(f) + \mathrm{supp}(g))$, which implies in particular

$$(\mathbb{R}^d \setminus (\mathrm{supp}(f) + \mathrm{supp}(g)))^\circ \subset (\mathbb{R}^d \setminus \mathrm{supp}(f * g))$$

and the assertion by taking complements of the above inclusion. □

We remark that if both f and g have compact support then so does $f * g$ (Exercise!).

3.53 definition. Let $\Omega \subset \mathbb{R}^d$ be open. Let $p \in [1, +\infty]$. The vector space $L^p_{loc}(\Omega)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that, for every compact subset $K \subset \Omega$, one has $f\mathbf{1}_K \in L^p(\Omega)$, or equivalently $f \in L^p(K)$.

3.54 exercise. Let $p \in [1, +\infty]$. Prove that if $f \in L^p$ than $f \in L^p_{loc}$. Show that the converse is not true in general. Prove that $L^q_{loc} \subset L^p_{loc}$ if $p \leq q$.

Note in particular that $L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$ for all $p \geq 1$.

We now start investigating on how convolutions inherit the regularity of just one of the two factors.

3.55 proposition. *Let $f \in C_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f * g$ is continuous on $\mathbb{R}^d$.*

Proof. Let $x_n \to x$. We first notice that

$$y \mapsto f(x-y)g(y)$$

has a well defined Lebesgue integral, because y ranges in the compact set $x - \mathrm{supp}(f)$ and g is summable on that set. Now, by possibly fattening the support of f, we can find a compact set K containing the set $x_n - \mathrm{supp}(f)$ for large enough n. Therefore, if $y \notin K$, then $x_n - y$ does not belong to the support of f and therefore $f(x_n - y) = 0$. Since f continuous, then f is uniformly continuous on its support. Hence, using that every uniformly continuous function f has a modulus of continuity

$$\omega(f)(\delta) = \sup\{|f(x) - f(y)| \,:\, |x-y| \le \delta\}$$

tending to zero as $\delta \searrow 0$, we get

$$|f(x_n - y) - f(x-y)| \le \omega(\delta_n)\mathbf{1}_K(y)\,,$$

as $|x_n - x| < \delta_n \searrow 0$. By integrating with respect to y we get

$$|(f*g)(x_n) - (f*g)(x)| \le \int_{\mathbb{R}^d} |g(y)||f(x_n-y) - f(x-y)|dy$$
$$\le \omega(\delta_n)\int_K |g(y)|dy$$

which proves the assertion since the last integral above is finite. □

3.56 proposition. *Let $f \in C^k_c\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f*g \in C^k(\mathbb{R}^d)$ and*

$$D^\alpha(f*g) = (D^\alpha f)*g$$

for any multi-index α with length less than k.

Proof. By induction we only need to prove the case $k=1$. Let $x \in \mathbb{R}^d$. We claim that $f*g$ is differentiable at x and that $\nabla(f*g)(x) = ((\nabla f)*g)(x)$. Let us fix $h \in \mathbb{R}^d$ with $|h| < 1$. For all $y \in \mathbb{R}^d$ we have

$$|f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)|$$
$$= \left|\int_0^1 (h\cdot\nabla f(x-y+sh) - h\cdot\nabla f(x-y))ds\right|\,.$$

Now, due to the uniform continuity of f and its first derivative on $\operatorname{supp}(f)$, the aboev integral can be controlled by $|h|\omega(h)$ for some modulus of continuity $\omega(h) \searrow 0$ as $|h| \searrow 0$. Let K be a compact set such that $x + B_1(0) - \operatorname{supp}(f) \subset K$. If $y \notin K$ then $x - y + h \notin \operatorname{supp}(f)$ for all h with $|h| < 1$. Therefore, for $y \notin K$ and $|h| < 1$,

$$f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y) = 0\,.$$

Therefore, similarly to the previous proposition

$$|f(x+h-y) - f(x-y) - h\cdot\nabla f(x-y)| \le |h|\omega(|h|)\mathbf{1}_K(y)\,.$$

Hence,

$$\begin{aligned}
&|(f*g)(x+h)-(f*g)(x)-h\cdot((\nabla f)*g)(x)| \\
&\leq \int_{\mathbb{R}^d} |g(y)||f(x+h-y)-f(x-y)-h\cdot\nabla f(x-y)|dy \\
&\leq |h|\omega(|h|)\int_K |g(y)|dy\,,
\end{aligned}$$

which implies the assertion by letting $|h|\searrow 0$. □

3.57 definition (Mollifiers). A sequence of *mollifiers* ρ_n is a sequence of functions on $\mathbb{R}^d$ such that

$$\rho_n \in C_c^\infty(\mathbb{R}^d)\,,\quad \operatorname{supp}(\rho_n)\subset \overline{B_{1/n}(0)}\,,\quad \int_{\mathbb{R}^d}\rho_n(x)dx = 1\,,\quad \rho_n\geq 0\,.$$

It is very easy to generate a family of mollifiers as follows. Take

$$\rho(x)=\begin{cases} e^{1/(|x|^2-1)} & \text{if } |x|<0 \\ 0 & \text{if } |x|\geq 1\,. \end{cases}$$

Then set $\rho_n(x)=Cn^d\rho(nx)$ with

$$C=\left(\int_{\mathbb{R}^d}\rho(x)dx\right)^{-1}\,.$$

3.58 proposition. *Assume $f\in C(\mathbb{R}^d)$. Then $\rho_n*f\to f$ uniformly on compact sets.*

Proof. Fix a compact set K in $\mathbb{R}^d$. Given $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x-y)-f(x)|<\varepsilon$$

provided $|y|<\delta$ and for all $x\in K$. Clearly, the δ depends on ε and on K. Now,

$$\begin{aligned}
(\rho_n*f)(x)-f(x) &= \int \rho_n(y)(f(x-y)-f(x))dy \\
&= \int_{B_{1/n}(0)} \rho_n(y)(f(x-y)-f(x))dy\,.
\end{aligned}$$

For $n>1/\delta$ and $x\in K$ we get

$$|(\rho_n*f)(x)-f(x)|\leq\varepsilon\int\rho_n(y)dy=\varepsilon\,.$$

□

3.59 theorem. *Assume $f \in L^p(\mathbb{R}^d)$ with $p \in [1, +\infty)$. Then $(\rho_n * f) \to f$ in L^p.*

Proof. Given $\varepsilon > 0$, we know there is a function $f_1 \in C_c(\mathbb{R}^d)$ with $\|f - f_1\|_{L^p} < \varepsilon$. We know that $\rho_n * f_1$ converges to f_1 uniformly on compact sets. On the other hand

$$\operatorname{supp}(\rho_n * f_1) \subset \overline{B_{1/n}(0)} + \operatorname{supp}(f_1) \subset \overline{B_1(0)} + \operatorname{supp}(f_1)\,,$$

which is a fixed compact set. Hence, it easily follows that

$$\|\rho_n * f_1 - f_1\|_{L^p} \to 0\,.$$

Now,

$$\rho_n * f - f = \rho_n * (f - f_1) + (\rho_n * f_1 - f_1) + (f_1 - f)\,,$$

which gives

$$\|\rho_n * f - f\|_{L^p} \leq 2\|f - f_1\|_{L^p} + \|\rho_n * f_1 - f_1\|_{L^p}\,,$$

as a consequence of Young's inequality for convolutions. Therefore,

$$\limsup_{n \to +\infty} \|\rho_n * f - f\|_{L^p} \leq 2\varepsilon$$

and the assertion follows from the arbitrariness of $\varepsilon > 0$. □

3.60 corollary. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. Postponed. □

3.61 proposition. *Let $u \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open. If $\int_\Omega u\phi dx = 0$ for all $\phi \in C_c^\infty(\Omega)$, then $u = 0$ almost everywhere on Ω.*

Proof. Let $g \in L^\infty(\mathbb{R}^d)$ be a function with compact support contained in Ω. Set $g_n = \rho_n * g$. Hence, for large enough n, $g_n \in C_c^\infty(\Omega)$. Hence, by assumption we have for all n large enough

$$\int u g_n dx = 0\,.$$

Since $g_n \to g$ in $L^1(\mathbb{R}^d)$, there exists a subsequence of g_n (still denoted by g_n for simplicity) converging to g almost everywhere on $\mathbb{R}^d$. Moreover, Young's inequality for convolutions implies $\|g_n\|_{L^\infty} \leq \|g\|_{L^\infty}$. Hence, by dominated convergence we get

$$\int u g dx = 0\,.$$

Let K be a compact subset of Ω and set

$$g(x) = \begin{cases} \operatorname{sign}(u(x)) & \text{if } x \in K \\ 0 & \text{otherwise}. \end{cases}$$

We deduce

$$0 = \int_{\mathbb{R}^d} ug dx = \int_K u \operatorname{sign}(u) dx = \int_K |u| dx .$$

Hence, $|u| = 0$ on K. Since K is arbitrary, $|u| = 0$ on Ω. □

3.6 *A criterion for strong compactness in L^p*

In this subsection we shall use the *shift function*

$$(\tau_h f)(x) = f(x+h) .$$

3.62 theorem (Kolmogorov-Riesz-Frechet). *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^d)$ with $1 \le p < +\infty$. Assume further that*

$$\lim_{|h| \searrow 0} \|(\tau_h f) - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \textit{uniformly on } f \in \mathcal{F} , \tag{38}$$

that is, for $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(\tau_h f) - f\|_{L^p(\mathbb{R}^d)} < \varepsilon$ for all $|h| < \delta$ and for all $f \in \mathcal{F}$. Then, $\mathcal{F}|_\Omega$ is relatively compact in $L^p(\Omega)$ for any measurable $\Omega \subset \mathbb{R}^d$ having finite measure.

Proof. The proof is performed in four steps.

Step 1. Under the assumptions above, we claim that

$$\|\rho_n * f - f\|_{L^p(\mathbb{R}^d)} \le \varepsilon$$

for all $f \in \mathcal{F}$ and for all $n > 1/\delta$. Indeed, Hoelder's inequality implies

$$\begin{aligned} |(\rho_n * f)(x) - f(x)| &\le \int \rho_n(y)|f(x-y) - f(x)|dy \\ &= \int \rho_n(y)^{1/p}\rho_n(y)^{1/p'}|f(x-y) - f(x)|dy \\ &\le \left(\int \rho_n(y)|f(x-y) - f(x)|^p dy\right)^{1/p} \left(\int \rho_n(y)dy\right)^{1/p'} \\ &= \left(\int \rho_n(y)|f(x-y) - f(x)|^p dy\right)^{1/p} . \end{aligned}$$

Hence,

$$\int |(\rho_n * f)(x) - f(x)|^p dx \le \iint \rho_n(y)|f(x-y) - f(x)|^p dy dx ,$$

and by assumption the above is controlled, for $n > 1/\delta$, by ε^p.

Step 2. We claim that there exists a constant C_n depending only on n such that

$$\|\rho_n * f\|_{L^\infty} \le C_n \|f\|_{L^p} \qquad \text{for all } f \in \mathcal{F}$$

and

$$|(\rho_n * f)(x_1) - (\rho_n * f)(x_2)| \le C_n \|f\|_{L^p} |x_1 - x_2| \qquad \text{for all } f \in \mathcal{F} \text{ and for all } x_1, x_2 \in \mathbb{R}^d .$$

Indeed, we have

$$|(\rho_n * f)(x)| \le \int \rho_n(y)|f(x-y)|dy \le \left(\int \rho_n(y)^{p'} dy\right)^{1/p'} \|f\|_{L^p}$$

and, since $\nabla(\rho_n * f) = (\nabla \rho_n) * f$, similarly we get

$$\|\nabla(\rho_n \ * f)\|_{L^\infty} \le \|\nabla \rho_n\|_{L^{p'}} \|f\|_{L^p} ,$$

which implies the assertion since the L^∞ norm of $\nabla(\rho_n * f)$ controls the difference quotients of $\rho_n * f$.

Step 3. Given $\varepsilon > 0$ and Ω with finite measure, we can find $\omega \subset \Omega$ bounded and measurable such that

$$\|f\|_{L^p(\Omega \setminus \omega)} < \varepsilon \qquad \text{for all } f \in \mathcal{F} .$$

Indeed, we write

$$\|f\|_{L^p(\Omega\setminus\omega)} \le \|f - (\rho_n * f)\|_{L^p(\mathbb{R}^d)} + \|\rho_n * f\|_{L^p(\Omega\setminus\omega)} ,$$

and the last term above is controlled by

$$m(\Omega \setminus \omega)^{1/p} C_n \|f\|_{L^p(\mathbb{R}^d)}$$

which can be made small by choosing $m(\Omega \setminus \omega)$ small, which is always possible since the measure of Ω is finite.

Step 4. Since $L^p(\Omega)$ is complete, to conclude we need to show that $\mathcal{F}|_\Omega$ is totally bounded. Let $\varepsilon > 0$. Let us fix $\omega \subset \Omega$ as above, and let us fix $n > 1/\delta$. The family $\mathcal{H} := (\rho_n * \mathcal{F})|_\omega$ satisfies all assumptions of Arzelà-Ascoli Theorem. Therefore, $\mathcal{H}$ is relatively compact in $C(\overline{\omega})$. Since ω has finite measure, it is easily checked that $\mathcal{H}$ has in fact compact closure in $L^p(\omega)$. Hence, by total boundedness we can cover $\mathcal{H}$ by a finite number of balls in $L^p(\omega)$ with radius ε. Therefore, there exists finitely many $g_i \in L^p(\omega)$, $i = 1, \dots, k$, such that

$$\mathcal{H} \subset \bigcup_{i=1}^{k} B_\varepsilon(g_i) .$$

Now, for all $i \in 1, \dots, k$ we set $\widetilde{g}_i : \Omega \to \mathbb{R}$ to be equal to g on ω and zero elsewhere. We claim that $\mathcal{F}|_\Omega$ can be covered by the balls of centers $\widetilde{g}_i$ and radius 3ε. Let $f \in \mathcal{F}$. There is some i such that

$$\|\rho_n * f - g_i\|_{L^p(\omega)} < \varepsilon\,.$$

Since

$$\|f - \widetilde{g}_i\|^p_{L^p(\Omega)} = \int\limits_{\Omega\setminus\omega} |f|^p dx + \int\limits_{\omega} |f - g_i|^p dx\,,$$

we have

$$\begin{aligned}&\|f - \widetilde{g}_i\|_{L^p(\Omega)} \le \varepsilon + \|f - g_i\|_{L^p(\omega)}\\ &\quad\le \varepsilon + \|f - \rho_n * f\|_{L^p(\mathbb{R}^d)} + \|\rho_n * f - g_i\|_{L^p(\omega)} \le 3\varepsilon\,.\end{aligned}$$

Hence, we conclude that $\mathcal{F}|\Omega$ can be covered by finitely many balls with radius 3ε, which implies total boundedness and the thesis. $\square$

3.63 remark. As a consequence of the previous theorem, let $\mathcal{F}$ be a bounded subset of $L^p(\mathbb{R}^d)$ with $p \in [1, +\infty)$ such that (38) holds and such that for every $\varepsilon > 0$ there exists a bounded set $\Omega \subset \mathbb{R}^d$ such that

$$\int\limits_{\mathbb{R}^d\setminus\Omega} |f|^p dx < \varepsilon \qquad \text{for all } f \in \mathcal{F}\,.$$

Then, $\mathcal{F}$ has compact closure in $L^p(\mathbb{R}^d)$.

In general, if we want to achieve strong compactness in L^p *on the whole space* $\mathbb{R}^d$, we need an additional assumption as in the previous remark, otherwise there may be situations in which just (38) alone (and the boundedness) is not sufficient for compactness, see the Exercises.

3.7 *Finite-dimensional Banach spaces*

In this subsection, we prove that every finite-dimensional (real or complex) normed linear space is continuous, and that all norms on a finite-dimensional space are equivalent. None of these statements is true for infinite-dimensional linear spaces. As a result, topological considerations can often be neglected when dealing with finite-dimensional spaces but are of crucial importance on infinite-dimensional psaces. we begin by proving that the components of a vector with respect to any basis of a finite-dimensional space can be bounded by the norm of the vector.

3.64 definition. Let X be a linear space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are *equivalent* if there are constants $c > 0$ and $C > 0$ such that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \qquad \text{for all } x \in X.$$

It is clear that if two norms are equivalent, then the two normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ have the same topology, i. e. they have the same convergent sequences (exercise).

Geometrically, two norms are equivalent if the unit ball of either one of the norms is contained in a ball of finite radius of the other norm.

3.65 lemma. *Let X be a finite-dimensional normed linear space with norm $\|\cdot\|$, and $\{e_1,\ldots,e_n\}$ any basis of X. There are constants $m>0$ and $M>0$ such that if $x = x_1e_1 + x_2e_2 + \ldots x_ne_n$, then*

$$m\sum_{i=1}^{n}|x_i| \le \|x\| \le M\sum_{i=1}^{n}|x_i|. \tag{39}$$

Proof. It suffices to prove the assertion for $x \in X$ such that $\|x\|_1 \doteq \sum_{i=1}^{n}|x_i| = 1$. Indeed, for a general $x\in X$, let $\tilde{x} = \frac{x}{\|x\|_1}$, we would have then $m \le \|\tilde{x}\| \le M$, i.e. $m\|x\|_1 \le \|x\| \le M\|x\|_1$. Now, the cube

$$C = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \ : \ \|x\|_1 = 1\}$$

is a closed, bounded subset of $\mathbb{R}^n$, and is therefore compact by the Heine-Borel theorem. We define a function $f : C \to X$ by

$$f((x_1,\ldots,x_n)) = \sum_{i=1}^{n} x_ie_i.$$

For $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in\mathbb{R}^n$, we have

$$\|f((x_1,\ldots,x_n)) - f((y_1,\ldots,y_n))\| \le \sum_{i=1}^{n}|x_i-y_i|\|e_i\| \le B\sum_{i=1}^{n}|x_i-y_i|,$$

where $B=\max_i\|e_i\|$, therefore f is continuous. Since $\|\cdot\| : X\to\mathbb{R}$ is continuous, the map

$$\mathbb{R}^n \ni (x_1,\ldots,x_n) \mapsto \|f((x_1,\ldots,x_n))\| \in \mathbb{R}$$

is continuous. Theorem 1.74 implies that $\|f\|$ is bounded on C and attains its infimum and supremum. Denoting the minimum by $m \ge 0$ and the maximum by $M \ge m$, we obtain the assertion except that we still have to prove that $m > 0$. Assume by contradiction that $m = 0$. This means that there exists $x_{\min}\in C$ such that $f(x_{\min}) = 0$. By definition of f this implies $x_{\min} = 0$, a contradiction because $x_{\min}\in C$. □

3.66 theorem. *Every finite-dimensional normed linear space is a Banach space.*

Proof. Suppose (x_k) is a Cauchy sequence in a finite-dimensional normed linear space X. Let $\{e_1,\ldots,e_n\}$ be a basis of X. We expand x_k as

$$x_k = \sum_{i=1}^{n} x_{k,i}e_i,$$

where $x_{i,k}\in\mathbb{R}$. For $1\le i\le n$, we consider the real sequence of i-th components, $(x_{k,i})_{k=1}^{+\infty}$. Equation (39) implies that

$$|x_{k,i} - x_{h,i}| \leq \frac{1}{m}\|x_k - x_h\|,$$

so $(x_{k,i})_{k=1}^{+\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is a $y_i \in \mathbb{R}$ such that

$$\lim_{k\to+\infty} x_{k,i} = y_i.$$

We define $y \in X$ by

$$y = \sum_{i=1}^{n} y_i e_i.$$

Then, from (39),

$$\|x_k - y\| \leq M \sum_{i=1}^{n} |x_{k,i} - y_i| \|e_i\|,$$

and hence $x_k \to y$ as $k \to +\infty$. Thus, every Cauchy sequence in X converges, and X is complete. □

As a consequence, we have the following corollary.

3.67 corollary. *Every finite-dimensional linear subspace of a normed linear space is closed.*

Finally, we show that although there are many different norms on a finite-dimensional linear space, they all lead to the same topology and the same notion of convergence.

3.68 theorem. *Any two norms on a finite-dimensional space are equivalent.*

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a finite-dimensional space X. We choose a basis $\{e_1, \ldots, e_n\}$ of X. then Lemma 3.65 implies that there are strictly positive constants m_1, m_2, M_1, M_2 such that if $x = \sum_{i=1}^{n} x_i e_i$ then

$$m_1 \sum_{i=1}^{n} |x_i| \leq \|x\|_1 \leq M_1 \sum_{i=1}^{n} |x_i|,$$
$$m_2 \sum_{i=1}^{n} |x_i| \leq \|x\|_2 \leq M_2 \sum_{i=1}^{n} |x_i|.$$

Hence, we have

$$\|x\|_1 \leq \frac{M_1}{m_1}\|x\|_2 \leq \frac{M_2}{m_1}\|x\|_1.$$

□

3.8 ℓ^p *spaces*

3.69 DEFINITION (ℓ_p spaces). Let $p \in [1, +\infty)$ be a real number. We say that a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is in ℓ_p if

$$\sum_{k=1}^{+\infty} |x_k|^p < +\infty.$$

The space of real sequences with the above property is called ℓ_p. For all $x \in \ell_p$, the quantity

$$\|x\|_{\ell_p} := \left[\sum_{k=1}^{+\infty} |x_k|^p\right]^{1/p}$$

is called the ℓ_p-*norm of* x. The space ℓ_∞ is the space of bounded sequences, i. e.

$$\sup_{k\in\mathbb{N}} |x_k| < +\infty.$$

For all $x \in \ell_\infty$, the quantity

$$\|x\|_{\ell_\infty} := \sup_{k\in\mathbb{N}} |x_k|$$

is called the ℓ_∞-*norm of* x.

The space ℓ_p can be seen as a subset of the vector space of all sequences of real numbers, with the obvious operations

- $x = \{x_k\}_k \in \ell_p$, $y = \{y_k\}_k \in \ell_p$, $x + y = \{x_k + y_k\}_k$
- $x = \{x_k\}_k \in \ell_p$, $\lambda \in \mathbb{R}$, $\lambda x = \{\lambda x_k\}_k$.

Proving that the sum between two vectors in well defined, as well as proving that $\|x\|_{\ell_p}$ is an actual norm, is not immediate. The first two properties of a norm (i. e. $\|x\|_{\ell_p} = 0$ implies $x = 0$, and $\|\lambda x\|_{\ell_p} = |\lambda|\|x\|_{\ell_p}$) are trivial. For the third one, i. e. the triangular inequality, we have to struggle a bit more. Once we have that, the sum between vectors will also be well defined, and we shall have a nice family of normed spaces to work with.

3.70 EXERCISE (Young's inequality). Let $p \in [1, +\infty)$ and let p' be its conjugate. Let $a, b \geq 0$ be two positive numbers. Then,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Solution. If $ab = 0$ there is nothing to prove. Assume $a > b > 0$. Set $A = a^p$ and $B = b^{p'}$. We need to prove that

$$A^{1/p} B^{1/p'} \leq \frac{A}{p} + \frac{B}{p'}.$$

Multiplication by $1/B$ makes the above inequality equivalent to

$$\frac{1}{p}\left(\frac{A}{B}\right)+\frac{1}{p'}\geq\left(\frac{A}{B}\right)^{1/p}$$

where we have used $\frac{1}{p'}-1=-\frac{1}{p}$. Now, set $t=\frac{A}{B}\geq 1$. The above becomes equivalent to proving that

$$\frac{1}{p}t+\frac{1}{p'}\geq t^{1/p}\qquad\text{for all } t\geq 1.$$

But the function $\phi(t)=\frac{1}{p}t+\frac{1}{p'}-t^{1/p}$ satisfies $\phi(1)=0$, $\phi'(t)=\frac{1}{p}-\frac{1}{p}t^{\frac{1}{p}-1}$, which is ≥ 0 for $t\geq 1$. Therefore, $\phi(t)\geq 0$ for all $t\geq 1$, which proves the assertion.

3.71 exercise (Discrete Hölder's inequality). Let $x=\{x_k\}_k\in\ell_p$ and $y=\{y_k\}_k\in\ell_{p'}$ with $p,p'\in[1,+\infty)$ conjugate numbers. Prove that

$$\sum_{k=1}^{+\infty}|x_k||y_k|\leq\|x\|_{\ell_p}\|y\|_{\ell_{p'}}.$$

Solution. Set

$$X_k=\frac{x_k}{\|x\|_{\ell_p}},\qquad Y_k=\frac{y_k}{\|y\|_{\ell_{p'}}},\qquad\text{for all } k\geq 1.$$

We need to prove that $\sum_{k=1}^{+\infty}|X_k||Y_k|\leq 1$. From Young's inequality, $|X_k||Y_k|\leq\frac{|X_k|^p}{p}+\frac{|Y_k|^{p'}}{p'}$ for all $k\geq 1$, and taking the sum over k we get

$$\sum_{k=1}^{+\infty}|X_k||Y_k|\leq\frac{1}{p}\sum_{k=1}^{+\infty}|X_k|^p+\frac{1}{p'}\sum_{k=1}^{+\infty}|Y_k|^{p'}=\frac{1}{p}+\frac{1}{p'}=1.$$

We are now ready to prove the triangular inequality on ℓ_p.

3.72 exercise (Discrete Minkowski's inequality). Let $x,y\in\ell_p\in[1,+\infty]$. Prove that $x+y\in\ell_p$ and $\|x+y\|_{\ell_p}\leq\|x\|_{\ell_p}+\|y\|_{\ell_p}$.

Solution. For $p<+\infty$, compute

$$\begin{aligned}\sum_{k\geq 1}|x_k+y_k|^p&=\sum_{k\geq 1}|x_k+y_k||x_k+y_k|^{p-1}\\&\leq\sum_{k\geq 1}|x_k||x_k+y_k|^{p-1}+\sum_{k\geq 1}|y_k||x_k+y_k|^{p-1},\end{aligned}$$

where we have used the obvious inequality $|x_k+y_k|\leq|x_k|+|y_k|$. Now, since

$p' = \frac{p}{p-1}$ is conjugate of p, the above discrete Hölder's inequality implies

$$\sum_{k\geq 1} |x_k + y_k|^p \leq \left(\sum_{k\geq 1} |x_k|^p\right)^{1/p} \left(\sum_{k\geq 1} |x_k + y_k|^p\right)^{\frac{p-1}{p}} + \left(\sum_{k\geq 1} |y_k|^p\right)^{1/p} \left(\sum_{k\geq 1} |x_k + y_k|^p\right)^{\frac{p-1}{p}},$$

which yields

$$\left(\sum_{k\geq 1} |x_k + y_k|^p\right)^{1-\frac{p-1}{p}} \leq \left(\sum_{k\geq 1} |x_k|^p\right)^{1/p} + \left(\sum_{k\geq 1} |y_k|^p\right)^{1/p},$$

which proves the assertion. The case $p = +\infty$ is a trivial exercise.

3.73 EXERCISE (Completeness of the ℓ_p spaces). Let $p \in [1, +\infty]$. Let $x_n = \{x_{n,k}\}_k$ be a Cauchy sequence in ℓ_p. Then, $x_n \to x$ as $n \to +\infty$ in $\|\cdot\|_{\ell_p}$ for some $x \in \ell_p$.

Solution.
Let us first consider the case $p = 1$. Since

$$\sum_{k\geq 1} |x_{n,k} - x_{m,k}| \to 0 \qquad \text{as } n, m \to +\infty,$$

for all $k \geq 1$ we have that $\{x_{n,k}\}_n$ is a Cauchy sequence in $\mathbb{R}$, and hence there exists some $x_k \in \mathbb{R}$ such that $x_{n,k} \to x_k$ as $n \to +\infty$. We need to prove that $x = \{x_k\} \in \ell_1$ and that $\|x_n - x\|_{\ell_1} \to 0$ as $n \to +\infty$. Let $\epsilon > 0$. The Cauchy condition on the sequence x_n reads

$$\sum_{k=1}^{+\infty} |x_{n,k} - x_{m,k}| < \epsilon \qquad \text{for } m \geq n \geq N_\epsilon,$$

for some $N_\epsilon \in \mathbb{N}$. Now let $K \in \mathbb{N}$, from above we have

$$\sum_{k=1}^{K} |x_{n,k} - x_{m,k}| < \epsilon \qquad \text{for } m \geq n \geq N_\epsilon,$$

and since $x_{n,k} \to x_k$ as $n \to +\infty$ (and the above sum has *finitely many* terms), we have

$$\sum_{k=1}^{K} |x_{n,k} - x_k| < \epsilon \qquad \text{for } n \geq N_\epsilon.$$

Since N_ϵ does not depend on K, we can take the supremum with respect to K above and get

$$\sum_{k\geq 1} |x_{n,k} - x_k| = \sup_{K\in\mathbb{N}} \sum_{k=1}^{K} |x_{n,k} - x_k| < \epsilon \qquad \text{for } n \geq N_\epsilon,$$

which shows that $\|x_n - x\|_{\ell_1} \to 0$ as $n \to +\infty$. By triangular inequality, we then get

$$\|x\|_{\ell_1} \le \|x - x_{N_\epsilon}\|_{\ell_1} + \|x_{N_\epsilon}\|_{\ell_1} \le \epsilon + \|x_{N_\epsilon}\|_{\ell_1},$$

and the last term above is finite.

Now, let $p \in (1, +\infty)$. Assume $\{x_n\}_n \in \ell_p$ is a Cauchy sequence. Hence, as above we can easily show that there exists a sequence $\{x_k\}_k$ of real numbers such that $x_{n,k} \to x_k$ as $n \to +\infty$ for all $k \ge 1$. To prove that $x = \{x_k\}_k \in \ell_p$, for a given ϵ let $N_\epsilon, k_\epsilon \in \mathbb{N}$ such that, for all $n, m \ge N_\epsilon$,

$$\sum_{k\ge 1} |x_{n,k} - x_{m,k}|^p < \epsilon, \qquad \text{for } m \ge n \ge N_\epsilon. \tag{40}$$

As before, for all $K \in \mathbb{N}$ we have

$$\sum_{k=1}^{K} |x_{n,k} - x_{m,k}|^p < \epsilon \qquad \text{for } m \ge n \ge N_\epsilon,$$

and the assertion $\|x_n - x\|_{\ell_p} \to 0$ follows similarly as in the case $p = 1$. The triangular inequality then proves once again that $x \in \ell_p$.

Finally, let us consider the case $p = +\infty$. Assume $\{x_n\}_n \in \ell_\infty$ is a Cauchy sequence. For a given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, for all $n, m \ge N_\epsilon$ one has

$$\sup_{k\in\mathbb{N}} |x_{n,k} - x_{m,k}| < \epsilon.$$

This implies that each sequence $\{x_{n,k}\}_n$ (for all $k \ge 1$) converges in n to some $x_k \in \mathbb{R}$. Let $x = \{x_k\}_k$. We can set $m > n$ and send $m \to +\infty$ above and get $|x_{n,k} - x_k| \le \epsilon$ for all $n \ge N_\epsilon$. This holds for all $k \ge 1$, hence $\|x_n - x\|_{\ell_\infty} \le \epsilon$ for all $n \ge N_\epsilon$. Moreover, $\|x\|_{\ell_\infty} \le \|x_{N_\epsilon} - x\|_{\ell_\infty} + \|x_{N_\epsilon}\|_{\ell_\infty}$, which proves that $x \in \ell_\infty$

3.9 *Exercises*

1. Let $A, B \subset \mathbb{R}^d$ be two Lebesgue measurable sets. Assume that $A = B \setminus C$ with C a measurable set with $m(C) = 0$. Then prove that $m(A) = m(B)$.

2. Find an example of a sequence of measurable functions f_n on $\mathbb{R}$ which do not satisfy the assumptions of Fatou's lemma and for which

$$\int \left(\liminf_{n\to+\infty} f_n(x) \right) dx > \liminf_{n\to+\infty} \int f_n(x)dx.$$

3. Show that the indicator function $\mathbf{1}_A$ of a set $A \subset \mathbb{R}^d$ is measurable if and only if the set A is Lebesgue measurable.

4. Suppose $\{f_n\}_n$ is a sequence of measurable, nonnegative functions. Assume $f_n \to f$ almost everywhere on $\mathbb{R}^d$. Prove that f is almost everywhere nonnegative.

5. Let $f : \mathbb{R}^d \to [0, +\infty]$ be a summable function. Show that for every $\epsilon > 0$ there exists a measurable set $E \subset \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d \setminus E} |f(x)| dx < \epsilon$$

(Hint: use the dominated convergence theorem).

6. For each of the following sequences of functions (restricted to the domain I),

   - determine whether or not they converge almost everywhere on I, and in the affirmative case find the almost everywhere limit f,
   - say whether or not $\int_I f_n dx \to \int_I f dx$ as $n \to +\infty$,
   - say whether or not the sequence converges uniformly to f on I:

   (a) $f_n(x) = \frac{nx}{1+n^2x^2}$, $I = [0, 1]$.

   (b) $f_n(x) = nxe^{-nx^2}$, $I = (0, 1)$.

   (c) $f_n(x) = \frac{n^2x^2}{n^4+x^2}$, $I = (1, +\infty)$.

   (d) $f_n(x) = \frac{nx}{1+n^2x^2}$, $I = (0, +\infty)$.

   (e) $f_n(x) = nxe^{-n^2x^2}$, $I = [0, 1]$.

   (f) $f_n(x) = nxe^{-n^2x^2}$, $I = [1, +\infty)$.

   (g) $f_n(x) = nxe^{-n^2x^2}$, $I = [0, +\infty)$.

   (h) $f_n(x) = \frac{1}{1+x^n}$, $I = [0, +\infty)$.

7. Let $E \subset \mathbb{R}^d$ be a measurable set. Let $f \in L^p(E)$ and $g \in L^q(E)$ for some $p, q \in [1, +\infty]$. Let $r \in [1, +\infty]$ be such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Prove that $f \cdot g \in L^r(E)$, and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

8. On $\mathbb{R}^d$, let

$$f_0(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and

$$f_\infty(x) = \begin{cases} |x|^{-\alpha} & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1. \end{cases}$$

Show that

- $f_0 \in L^p$ if and only if $p\alpha < d$.
- $f_\infty \in L^p$ if and only if $p\alpha > d$.

9. For each of the following functions defined on the set $I \subset \mathbb{R}$, say for which $p \in [1, +\infty]$ the function f belongs to $L^p(I)$:

    (a) $f(x) = \frac{\sin|x|}{|x|^2}$, $I = [-1, 1]$.

    (b) $f(x) = x^2/3 \log x$, $I = (0, 1)$.

    (c) $f(x) = \sqrt{\frac{x^3}{1+x^2+x^4}}$, $I = [0, +\infty)$.

    (d) $f(x) = \frac{\arctan x}{x}$, $I = (0, +\infty)$.

    (e) $f(x) = \frac{1}{x \log x}$, $I = (0, 1)$.

    (f) $f(x) = \frac{1}{x \log x}$, $I = (1, +\infty)$.

10. For each of the following sequences of functions defined on the set $I \subset \mathbb{R}$,

    - determine whether or not they converge almost everywhere on I, and in the affirmative case find the almost everywhere limit f,
    - say whether or not $f_n \to f$ in $L^p(I)$ for the index p indicated:

    (a) $f_n(x) = \frac{n^2x^2}{1+n^3x^3}$, $I = [0, +\infty)$, $p = 2$,

    (b) $f_n(x) = \frac{1}{1+nx^{1/3}}$, $I = [0, +\infty)$, $p = 2$,

    (c) $f_n(x) = n\sin(x/n)e^{-2x}$, $I = [0, +\infty)$, $p = 1$,

    (d) $f_n(x) = \frac{1}{1+n\sqrt{x}}$, $I = [0, 1]$, $p = 1$.

    (e) $f_n(x) = ne^{-nx}$, $I = (0, 1)$, $p \in [1, +\infty)$.

    (f) $f_n(x) = n^{1/3}e^{-nx}$, $I = (0, +\infty)$, $p \in [1, +\infty]$

11. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined via

    $$f_n(x) \begin{cases} -1 & \text{if } x < -1/n \\ n & \text{if } -1/n \leq x \leq 1/n \\ 1 & \text{if } x > 1/n. \end{cases}$$

    - Find the almost everywhere limit of f_n as $n \to +\infty$.
    - Prove that f_n does not converge uniformly to f.
    - Prove that f_n converges to f in $L^p(\mathbb{R})$ if and only if $p \in [1, +\infty)$.

12. Let $\mathcal{B}$ be a bounded subset of $L^p(\mathbb{R}^d)$ with p finite and let $G \in L^1(\mathbb{R}^d)$. Consider the set

    $$\mathcal{F} = \{G * f \;:\; f \in \mathcal{B}\} .$$

    Prove that $\mathcal{F}|_\Omega$ has compact closure in $L^p(\Omega)$ for any measurable set Ω with finite measure.

13. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ and let

$$\mathcal{F} = \{\psi_n(x) = \varphi(x+n) \, : \, n \in \mathbb{N}\} \, .$$

Prove that $\mathcal{F}$ satisfies (38) but it doesn't have compact closure in $L^p(\mathbb{R}^d)$ for p finite.

3.10 *Envisaged outcomes*

At the end of this chapter, the student should be familiar with

- The main differences between Peano-Jordan integration theory and Lebesgue integration theory.
- The definition and the main properties of Lebesgue measure and integration.
- The three main theorems regarding the limit interchange properties of Lebesgue integration, the main examples in which that property fails. The student should be able to determine whether or not such property holds in the exercises.
- The definition and the main properties of L^p spaces, including the main integral inequalities (Hölder's and Minkowski's), completeness, density properties, and separability. The student must be able to determine whether or not a given function belongs to a given L^p space.
- The notion of convergence in L^p. The student should be able to establish that a sequence converges (or does not converge) in L^p in the exercises.
- The concept of strong compactness in L^p, the use of Kolmogorov-Riesz-Frechet theorem.

4 Introduction to linear operators on Banach spaces

Many linear equations can be formulated in terms of a suitable linear operator acting on a Banach space (see Problem 0.2). In this section we study linear operators acting on Banach spaces in greater detail.

4.1 *Bounded linear maps*

We recall the concept of *linear operator*, which should be well known from basic linear algebra. A *linear map* or *linear operator* T between real (or complex) linear spaces X, Y is a function $T : X \to Y$ such that

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty, \quad \text{for all } \lambda, \mu \in \mathbb{R} \text{ (or } \mathbb{C}\text{) and } x, y \in X.$$

A linear map $T : X \to X$ is called a *linear transformation* of X, or a linear operator on X. If a linear map $T : X \to Y$ is one-to-one and onto, then we say that T is *invertible*, and define the inverse map $T^{-1} : Y \to X$ by $T^{-1}y = x$ if and only if $Tx = y$, so that $TT^{-1} = \mathbb{I}_Y$, $T^{-1}T = \mathbb{I}_X$. The linearity of T implies the linearity of T^{-1} (exercise!). An useful exercise in this contexts is to prove that for a linear operator T, the image of the zero vector in X, $T(0)$, is always equal to the zero vector in Y.

A natural question arises: are linear operators *continuous*? In the finite dimensional case we expect this is the case. Sadly, in infinite dimension this is not always the case.

4.1 PROPOSITION. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. Let $T : X \to Y$ be a linear map. Then, the following are equivalent:*

(i) T is continuous at $x = 0 \in X$.

(ii) T is continuous on all points $x \in X$.

(iii) T maps bounded sets of X into bounded sets of Y.

(iv) There exists $M > 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X. \tag{41}$$

Proof. (i) implies (ii): Assume T is continuous at $0 \in X$. Let $x \in X$. Let $\{x_n\}_n$ be a sequence in X which converges to x as $n \to +\infty$. Consider

$$\|T(x_n) - T(x)\|_Y = \|T(x_n - x)\|_Y \to 0,$$

because $x_n - x$ converges to zero and T is continuous at zero.

(ii) implies (iii): Let $A \subset X$ be a bounded set. This means $A \subseteq B_R(0)$ for some $R \geq 0$. We claim there exists $S \geq 0$ such that $T(A) \subseteq B_S(0) \subset Y$. Assume by contradiction that for every $n \in \mathbb{N}$ there exists $x_n \in A$ such that $\|T(x_n)\|_Y \geq n$. Set $v_n := \frac{x_n}{n}$. Since $\|x_n\|_X \leq R$, then $v_n \to 0$ as $n \to +\infty$. On the other hand,

$$\|T(v_n)\|_Y = \left\| \frac{T(x_n)}{n} \right\|_Y = \frac{1}{n}\|T(x_n)\|_Y \geq 1.$$

This is a contradiction with the fact that T is continuous at zero.

(iii) implies (iv): By contradiction, assume that for all $n \in \mathbb{N}$ there exists $x_n \in X$ with $\|T(x_n)\|_Y \geq n\|x_n\|_X$. Set $y_n = \frac{x_n}{\|x_n\|_X}$. Clearly $\|y_n\|_X = 1$. Moreover,

$$\|T(y_n)\|_Y = \frac{1}{\|x_n\|_X}\|T(x_n)\|_Y \geq n,$$

which implies the set $T(\{y_n\}_n)$ is unbounded, whereas $\{y_n\}_n$ is bounded, a contradiction.

(iv) implies (i): Let $\{x_n\}_n$ converge to zero as $n \to +\infty$. Then, the condition $\|T(x)\|_Y \leq M\|x\|_X$ easily implies $\|T(x_n)\|_Y \to 0$, which gives the continuity of T at zero. □

4.2 definition. Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called a *bounded operator* if it is continuous. The (operator) *norm of* T is the number

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X},$$

i. e. $\|T\|$ is the infimum of all M such that condition (41) is satisfied. The space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$.

4.3 exercise. Let $T : X \to Y$ be linear and bounded and let $\|T\|$ be its norm. Then,

$$\|T\| = \inf\{M \geq 0 \,:\, \|Tx\| \leq M\|x\|\,, \text{ for all } x \in X\}\,.$$

Moreover,

$$\|T\| = \sup_{x\neq 0 \ \|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|\,.$$

To prove the latter, we observe first of all that

$$\frac{\|Tx\|}{\|x\|} = \left\| T\frac{x}{\|x\|} \right\|,$$

which shows that $\|T\|$ is the supremum of $\|Tz\|$ on the set $\|z\| = 1$. Then, for all $x \in X$ with $\|x\| \leq 1$ we set $\widetilde{x} = \frac{x}{\|x\|}$ and observe

$$\|T\widetilde{x}\| = \frac{\|Tx\|}{\|x\|} \geq \|Tx\|$$

which implies the supremum of $\|Tx\|$ on $\|x\| \leq 1$ is bounded from above by the supremum of $\|Tx\|$ on $\|x\| = 1$. The opposite inequality is trivial.

4.4 exercise. With the notation in the previous definition, prove that $\|T(x)\|_Y \leq \|T\|\|x\|_X$.

4.5 exercise. Prove that the operator norm $\|\cdot\|$ is a norm on the linear space $\mathcal{L}(X,Y)$.

4.6 proposition. *Let X be a normed space and Y be a Banach space. Then the space $\mathcal{L}(X,Y)$ is a Banach space*

Proof. We only need to prove that $\mathcal{L}(X,Y)$ is complete. Let T_n be a Cauchy sequence on $\mathcal{L}(X,Y)$ with respect to the operator norm $\|\cdot\|$. This means that, for all $\epsilon > 0$, there exists an $N(\epsilon)$ such that $\|T_n - T_m\| \leq \epsilon$ for all $n, m \geq N(\epsilon)$. Hence, for all $x \in X$,

$$\|T_n(x) - T_m(x)\|_Y = \|(T_n - T_m)(x)\|_Y \leq \|T_n - T_m\|\|x\|_X \leq \epsilon\|x\|_X.$$

The above shows that the sequence $T_n(x)$ is a Cauchy sequence in Y, which is complete, i. e. there exists an element $y \in Y$ such that $T_n(x) \to y$. Since y depends on x via T, we name $y = T(x)$. It is easily shown that T is a linear operator. Indeed, for $x_1, x_2 \in X$, we have

$$T(x_1 + x_2) = \lim_{n\to+\infty} T_n(x_1 + x_2) = \lim_{n\to+\infty} (T_n(x_1) + T_n(x_2)) = T(x_1) + T(x_2).$$

Moreover, T is a bounded operator. To see this, let $\epsilon > 0$. Then, one easily sees that there exists $N \in \mathbb{N}$ such that $\|T - T_N\| \leq \epsilon$ (as a consequence of the Cauchy condition on T_n). Therefore,

$$\|T(x)\|_Y = \|(T - T_N)(x)\|_Y + \|T_N(x)\|_Y \leq \|T - T_N\|\|x\|_X + \|T_N\|\|x\|_X \leq \epsilon\|x\|_X + M_N\|x\|_X,$$

where $M_N = \|T_N\|$. This implies $\|T(x)\|_Y \leq (\epsilon + M_N)\|x\|_X$, i. e. T is a bounded operator. □

4.7 definition. We say that a sequence $T_n \in \mathcal{L}(X,Y)$ is *norm-convergent* to $T \in \mathcal{L}(X,Y)$ if $\|T_n - T\| \to 0$ as $n \to +\infty$. We say that T_n converges to T *pointwise* if $\|T_n(x) - T(x)\|_Y \to 0$ as $n \to +\infty$ for all $x \in X$.

4.8 exercise. Prove that $T : X \to Y$ is bounded if and only if T maps the unit ball $\{\|x\| \leq 1\}$ into a bounded set.

4.9 example. The linear map $A : \mathbb{R} \to \mathbb{R}$ defined by $Ax = ax$, where $a \in \mathbb{R}$, is bounded, and has norm $\|A\| = |a|$.

4.10 example. The identity map $\mathbb{I} : X \to X$ is bounded on any normed space X, and has norm one. If a map has norm zero, then it is the zero map $0x = 0$.

4.11 theorem. *Every linear operator on a finite-dimensional linear space is bounded.*

Proof. Suppose that $A : X \to X$ is a linear map and X is finite dimensional.

Let $\{e_1, \dots, e_n\}$ be a basis of X. If $x = \sum_{i=1}^n x_i e_i \in X$, then (39) implies that

$$\|Ax\| \leq \sum_{i=1}^{n} |x_i| \|Ae_i\| \leq \max_{1\leq i\leq n} \{\|Ae_i\|\} \sum_{i=1}^{n} |x_i| \leq \frac{1}{m} \max_{1\leq i\leq n} \{\|Ae_i\|\} \|x\|,$$

so A is bounded. □

Linear maps on infinite-dimensional normed spaces need not be bounded.

4.12 example. Let $X = C^\infty([0,1])$ consist of the smooth functions on $[0,1]$ that have continuous derivatives of all orders. equipped with the maximum norm $\|\cdot\|_\infty$. The space X is a normed space, but is not a Banach space, since it is incomplete. The differentiation operator $Du = u'$ is an unbounded linear map $D : X \to X$. For example, the function $u(x) = e^{\lambda x}$ satisfies $Du = \lambda u$. Thus, $\|Du\|/\|u\| = |\lambda|$ may be arbitrarily large. The unboundedness of differential operators is a fundamental difficulty in their study.

4.13 exercise (Operator norms of finite dimensional matrices). We now show how, in the finite dimensional case, the norm of an operator cam be computed depending on norm on the space, in terms of the associated matrix. Suppose that $A : X \to Y$ is a linear map between finite-dimensional real linear spaces X, Y with $\dim X = n$, $\dim Y = m$. We choose bases $\{e_1, e_2, \dots, e_n\}$ of X and $\{f_1, f_2, \dots, f_m\}$ of Y. Then

$$A(e_j) = \sum_{i=1}^{m} a_{ij} f_i,$$

for a suitable $m \times n$ matric (a_{ij}) with real entries. We expand $x \in X$ as

$$x = \sum_{i=1}^{n} x_i e_i,$$

where $x_i \in \mathbb{R}$ is the i-th component of x. It follows from the linearity of A that

$$A(x) = A\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j A(e_j) = \sum_{i=1}^{m} y_i f_i, \qquad y_i = \sum_{j=1}^{n} x_j a_{ij}.$$

Thus, given a choice of bases for X, Y, we may represent A as a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$, where

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{42}$$

We will often use the same notation A to denote a linear map on a finite-dimensional space and its associated matrix, but is important not to confuse the geometrical notion of a linear map with the matrix of numbers that represents it.

Each pair of norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ induces a corresponding operator norm (or matrix norm) on A. We first consider the *Euclidean nor*m, or 2-norm, $\|A\|_2$ of A. The Euclidean norm of a vector x is given by $\|x\|_2 = (x, x)$, where $(x, y) = x^T y$. We may compute the Euclidean norm of A by maximizing $\|Ax\|_2$ on the unit sphere $\|x\|_2 = 1$. The maximizer x is a critical point of the function

$$f(x, \lambda) = (Ax, Ax) - \lambda\{(x, x) - 1\},$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Computing ∇f and setting it equal to zero, we find that x satisfies

$$A^T Ax = \lambda x. \tag{43}$$

Hence, x is an eigenvector of the matrix $A^T A$ and λ is an eigenvalue. The matrix $A^T A$ is an $n \times n$ symmetric matrix, with real, nonnegative eigenvalues (this easily follows after multiplying (43) by x via scalar product). At an eigenvector x of $A^T A$ that satisfies (43), normalized so that $\|x\|_2 = 1$, we have $(Ax, Ax) = \lambda$. Thus, the maximum value of $\|Ax\|_2$ on the unit sphere is the maximum eigenvalue of $A^T A$.

We define the *spectral radius* $r(B)$ of a squared matrix B to be the maximum absolute value of its eigenvalues. It follows that the Euclidean norm of A is given by

$$\|A\|_2 = \sqrt{r(A^T A)}. \tag{44}$$

In the case of linear maps $A : \mathbb{C}^n \to \mathbb{C}^m$ on finite dimensional complex linear spaces, equation (44) holds with A^T replaced by A^*, where A^* is the Hermitian conjugate of A.

To compute the maximum norm of A, we observe from (42) that

$$|y_i| \leq |a_{i1}||x_1| + |a_{i2}||x_2| + \ldots + |a_{in}||x_n| \leq (|a_{i1}| + \ldots + |a_{in}|)\|x\|_\infty.$$

Taking the maximum of this equation with respect to i and comparing the result with the definition of operator norm, we conclude that

$$\|A\|_\infty \leq \max_{1 \leq i \leq m} (|a_{i1}| + \ldots + |a_{in}|).$$

Conversely, suppose that the maximum on the right-hand side of this equation is attained at $i = i_0$. Let x be the vector with components $x_j = \text{sign } a_{i_0 j}$, where sign is the sign function

$$\text{sign } x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then, if A is nonzero, we have $\|x\|_\infty = 1$, and

$$\|Ax\|_\infty = |a_{i_0 1}| + \ldots + |a_{i_0 n}|,$$

which shows that

$$\|A\|_\infty = \max_{1\le i\le m}\left(\sum_{j=1}^{n} |a_{ij}|\right).$$

A similar argument (exercise) shows that the sum norm of A is given by the maximum column sum

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{m} |a_{ij}|.$$

For $1 < p < +\infty$, one can show (we omit the proof) that

$$\|A\|_p \le \|A\|_1^{1/p}\|A\|_\infty^{1-1/p}.$$

There are norms on the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{n\times n}$ of $m \times n$ matrices that are not associated with any vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$. An example is the *Hilbert-Schmidt norm*

$$\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$

Next, we give some examples of linear operators on infinite-dimensional spaces.

4.14 example. Let $X = \ell^\infty(\mathbb{N})$ be the space of bounded sequences $x = \{(x_1, x_2, \ldots)\}$ with the norm

$$\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|.$$

A linear map $A : X \to X$ is represented by an infinite matrix $(a_{ij})_{i,j=1}^{+\infty}$, where

$$(Ax)_i = \sum_{j=1}^{+\infty} a_{ij}x_j.$$

In order for this sum to converge for any $x \in \ell^\infty(\mathbb{N})$, we require that

$$\sum_{j=1}^{+\infty} |a_{ij}| \le +\infty$$

for each $i \in \mathbb{N}$, and in order for Ax to belong to $\ell^\infty(\mathbb{N})$, we require that

$$\sup_{i \in \mathbb{N}} \left(\sum_{j=1}^{+\infty} |a_{ij}| \right) < +\infty.$$

Then A is a bounded linear operator on $\ell^\infty(\mathbb{N})$, and its norm is the maximum row sum

$$\|A\|_\infty = \sup_{i \in \mathbb{N}} \left(\sum_{j=1}^{+\infty} |a_{ij}| \right).$$

The details are omitted.

4.15 EXAMPLE. Let $X = \ell^p(\mathbb{N})$. Consider the operator $T : X \to X$ defined by

$$(Tx)_n = \alpha_n x_n,$$

for all $x \in \ell^p$, with $\alpha = (\alpha_n)_n$ a given sequence of real numbers. Let us figure out what condition we should impose on α_n in order to have T bounded from X into itself, and let us compute the operator norm. A simple estimate gives

$$\|Tx\|_{\ell^p}^p \leq \sup_n |\alpha_n| \|x\|_{\ell^p}^p.$$

The above estimate is sharp: let α_{n_k} be a subsequence converging to $\|\alpha\|_{\ell^\infty}$ as $k \to +\infty$. Let x^k be the sequence in ℓ^p defined by

$$(x^k)_i = \delta_{i,n_k}, \qquad i = 1, 2, 3, \ldots.$$

We immediately see that $\|Tx^k\|_{\ell^p}^p$ tends to $\|\alpha\|_{\ell^\infty}$ ¡as $k \to +\infty$ and $\|x^k\|_{\ell^p} = 1$. Therefore,

$$\|T\| = \|\alpha\|_{\ell^\infty}.$$

4.16 EXAMPLE. Let $X = C([0,1])$ with the maximum norm, and

$$k : [0,1] \times [0,1] \to \mathbb{R}$$

be a continuous function. We define the linear Fredholm integral operator $K : X \to X$ by

$$Kf(x) = \int_0^1 k(x,y) f(y) dy.$$

Then K is bounded and

$$\|K\| \leq \max_{0 \leq x \leq 1} \left(\int_0^1 |k(x,y)| dy \right).$$

The details are left as an exercise.

4.2 *The kernel and range of a linear map*

The kernel and range are two important linear subspaces associated with a linear map.

4.17 DEFINITION. Let $T : X \to Y$ be a linear map between linear spaces X, Y. The *null space* or *kernel* of T, denoted by $\ker T$, is the subset of X defined by

$$\ker T = \{x \in X : \; Tx = 0\}.$$

The range of T, denoted by $\mathrm{Ran}T$, is the subset of Y defined by

$$\mathrm{Ran}T = \{y \in Y : \text{ there exists } x \in X \text{ such that } Tx = y\}.$$

The word 'kernel' is also used in a completely different sense to refer to the kernel of an integral operator. A linear map $T : X \to Y$ is one-to-one if and only if $\ker T = \{0\}$, and is onto if and only if $\mathrm{Ran}T = Y$.

4.18 EXERCISE. Let $T : X \to Y$ be a linear map between linear spaces X, Y. Prove that $\ker T$ is a linear subspace of X and $\mathrm{Ran}T$ is a linear subspace of Y. If X and Y are normed linear spaces and T is bounded, prove that the kernel of T is a closed linear subspace of X.

The *nullity* of T is the dimension of the kernel of T, and the *rank* of T is the dimension of the range of T. We now consider some examples.

4.19 EXAMPLE. The right shift operator on $\ell^\infty(\mathbb{N})$ is defined by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots),$$

and the left shift operator T by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

These maps have norm one (exercise!). Their matrices are the infinite-dimensional Jordan blocks

$$[S] = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad [T] = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The kernel of S is $\{0\}$ and the range of S is the subspace

$$\mathrm{Ran}S = \{(0, x_2, x_3, \dots) \in \ell^\infty(\mathbb{N})\}.$$

The range of T is the whole space $\ell^\infty(\mathbb{N})$, and the kernel of T is the one-dimensional subspace

$$\ker T = \{(x_1, 0, 0, \dots) \ : \ x_1 \in \mathbb{R}\}.$$

The operator S is one-to-one but not onto, and T is onto but not one-to-one. This cannot happen for linear maps $T : X \to X$ on a finite-dimensional space X, such as $X = \mathbb{R}^n$. In that case, $\ker T = \{0\}$ if and only if $\mathrm{Ran} T = X$.

4.20 example. Let $X = C([0,1])$ with the sup norm. We define the integral operator $K : X \to X$ by

$$Kf(x) = \int_0^x f(y) dy. \tag{45}$$

An integral operator like this one, with a variable range of integration, is called a *Volterra integral operator*. Then, K is bounded, with $\|K\| \leq 1$, since

$$\|Kf\|_\infty \leq \sup_{0 \leq x \leq 1} \int_0^x |f(y)| dy \leq \int_0^1 |f(y)| dy \leq \|f\|_\infty.$$

In fact, $\|K\| = 1$, since $K(1) = x$, and $\|x\|_\infty = 1$. The range of K is the set of continuously differentiable functions on $[0,1]$ that vanish at $x = 0$. This is a linear subspace of $C([0,1])$ but it is not closed. The lack of closure of the range of K is due to the smoothing effect of K, which maps continuous functions to differentiable functions.

4.21 theorem. *Let X, Y, Z be normed linear spaces. If $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, then $ST \in \mathcal{L}(X,Z)$, and*

$$\|ST\| \leq \|S\| \|T\|.$$

Proof. Exercise. □

4.22 example. Consider the linear maps A, B on $\mathbb{R}^2$ with matrices

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}.$$

These matrices have the Euclidean (or sum, or maximum) norms $\|A\| = \lambda$ and $\|B\| = \mu$, but $\|AB\| = 0$.

4.23 example. Let $X = C([0,1])$ equipped with the supremum norm. For $k_n(x,y)$ a real-valued continuous function on $[0,1] \times [0,1]$, we define $K_n \in \mathcal{L}(X)$ by

$$K_n f(x) = \int_0^1 k_n(x,y) f(y) dy.$$

Then $K_n \to 0$ in norm as $n \to +\infty$ if

$$\|K_n\| = \max_{x\in[0,1]} \int_0^1 |k_n(x,y)|dy \to 0 \qquad \text{as } n \to +\infty.$$

As example satisfying the above condition, take $k_n(x,y) = xy^n$.

4.3 *Compact operators*

A particularly important class of bounded operators is the class of compact operators.

4.24 definition. A linear operator $T : X \to Y$ is *compact* if $T(B)$ is a precompact subset of Y for every bounded subset B of X.

An equivalent formulation is that T is compact if and only if every bounded sequence (x_n) in X has a subsequence $(x_{n_k})_k$ such that $(Tx_{n_k})_k$ converges in Y. We do not require the range of T to be closed, so $T(B)$ need not be compact even if B is a closed bounded set. Another equivalent formulation is that T is compact if and only if T maps the closed unit ball $\{\|x\| \leq 1\}$ of X into a precompact subset of Y.

4.25 example. We propose immediately a classical example of a compact linear operator on an infinite dimensional Banach space. Let $X = C([0,1])$ and consider the operator $T \in \mathcal{L}(C)$ defined by

$$(Tf)(x) = \int_0^x f(y)dy,$$

called *Volterra operator*. It is an easy exercise to verify that T is linear and bounded. Now, let $f \in X$ be in a bounded set $B \subset X$. Since B is bounded, there is a constant M such that $\|f\|_\infty \leq M$ for all $f \in B$. Now, for all $f \in B$, we have that Tf is differentiable. Moreover,

$$\|(Tf)'\|_\infty \leq \sup_{x\in[0,1]} \int_0^x |f(y)|dy \leq \|f\|_\infty \leq M.$$

Since $(Tf)(0) = 0$ for all $f \in B$, we can use example 2.17 (consequence of Arzelá-Ascoli) to show that $T(B)$ is precompact. Hence, T is a compact operator.

We leave the proof of the following properties of compact operators as an exercise.

4.26 proposition. *Let X, Y, Z be Banach spaces.*

(a) If $S, T \in \mathcal{L}(X,Y)$ are compact, then any linear combination of S and T is compact.

(b) *Let $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$. If S is bounded and T is compact, or if S is compact and T is bounded, then $TS \in \mathcal{L}(X, Z)$ is compact.*

(c) *If T is bounded and* $\operatorname{Ran} T$ *has finite dimension, then T is compact. In this case we say that T is a* finite-rank operator.

4.27 example. If (T_n) is a sequence of compact operators in $\mathcal{L}(X, Y)$ converging uniformly to T, then T is compact. To see this, let $\epsilon > 0$ and let $N_\epsilon \in \mathbb{N}$ such that $\|T - T_{N_\epsilon}\| < \epsilon/2$. Let B be the closed unit ball of X. Since $T_{N_\epsilon}(B)$ is precompact, then $\overline{T_{N_\epsilon}(B)}$ is totally bounded. Hence, there exists $y_1, \ldots, y_{M_\epsilon} \in \overline{T_{N_\epsilon}(B)}$ such that $\overline{T_{N_\epsilon}(B)} \subset \bigcup_{i=1}^{M_\epsilon} B_{\epsilon/2}(y_i)$. Therefore, for a given $x \in B$ there exists $i \in \{1, \ldots, M_\epsilon\}$ such that $\|T_{N_\epsilon} x - y_i\| < \epsilon/2$. Hence,

$$\|Tx - y_i\| \leq \|Tx - T_{N_\epsilon} x\| + \|T_{N_\epsilon} x - y_i\| < \epsilon.$$

This proves that $T(B)$ is totally bounded, i. e. $T(B)$ is precompact.

As a consequence of the previous example, if X ad Y are Banach spaces the space $\mathcal{K}(X, Y)$ of compact linear operators from X into Y is a closed linear subspace of $\mathcal{L}(X, Y)$. Moreover, if (T_n) is a sequence of finite-rank operators converging uniformly to T, then T is a compact operator. The converse is also true for compact operators on many Banach spaces, including Hilbert spaces, although there exists separable Banach spaces on which some compact operators cannot be approximated with finite-rank operators.

We conclude this section with some important considerations on the compactness of the unit ball on infinite dimensional spaces. The compact sets of the Euclidean space $\mathbb{R}^d$ are characterised as those sets which are closed and bounded, according to Heine-Borel's theorem. This property is valid for all finite dimensional normed spaces. Indeed, this property *characterises* finite dimensional normed spaces, i. e. *if the dimension of the space is infinite such property is no longer true*. This is seen more precisely in the next theorem. First we prove the following Lemma.

4.28 lemma (Riesz's lemma). *Let $(E, \|\cdot\|)$ be a normed space. Let $M \subset E$ be a proper closed linear subspace of E. Let $\alpha \in (0, 1)$. Then, there exists a vector $x_\alpha \in E$ such that $\|x_\alpha\| = 1$ and $\inf\{\|x - x_\alpha\|,\ x \in M\} > \alpha$.*

Proof. For a given $y \in E \setminus M$, let $d = \inf_{x \in M} \|x - y\|$. Since M is closed, $d > 0$. Indeed, if $d = 0$ there would be no open balls centered on x entirely contained in $E \setminus M$, which would contradict $E \setminus M$ being an open set. Hence, since $\alpha \in (0, 1)$, there exists a vector $x_0 \in M$ with $d \leq \|y - x_0\| \leq \frac{d}{\alpha}$. Set $x_\alpha = \frac{y - x_0}{\|y - x_0\|}$. Clearly $\|x_\alpha\| = 1$. Moreover, for all $x \in M$ we have

$$\|x - x_\alpha\| = \left\| x - \frac{y - x_0}{\|y - x_0\|} \right\| = \left\| \frac{\|y - x_0\| x + x_0 - y}{\|y - x_0\|} \right\| \geq \frac{\alpha}{d} \| \|y - x_0\| x + x_0 - y\|.$$

Since $\|y - x_0\| x + x_0 \in M$, the quantity $\| \|y - x_0\| x + x_0 - y\|$ is bigger than d, and therefore $\|x - x_\alpha\| \geq \alpha$. □

We are now ready to prove the next important theorem.

4.29 THEOREM. *Let $(E, \|\cdot\|)$ be a normed space, and let $\overline{B_1(0)}$ be the closed unit ball of E. Then, $\overline{B_1(0)}$ is compact if and only if the dimension of E is finite.*

Proof. Assume first that E is finite dimensional, let $\{e_1, \ldots, e_d\}$ be a basis for E. Let us set for $x = \sum_{i=1}^d x_i e_i$, $T(x) = (x_1, \ldots, x_d) \in \mathbb{R}^d$. The linear map $T : E \to \mathbb{R}^d$ is a homeomorphism. Then $\sup_{x \in B_1(0)} \|x\|_2 < +\infty$, where $\|x\|_2 = \left(\sum_{i=1}^d |x_i|^2\right)^{1/2}$. The norm $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^d$. Hence, we can apply Heine-Borel theorem on $\mathbb{R}^d$: since the set $\overline{\{T(x)\}_{x \in B_1(0)}}$ is closed and bounded in the Euclidean norm, Heine-Borel theorem gives that the set is compact. Since T^{-1} is continuous, $\overline{B_1(0)}$ is also compact.

Now, assume that $B_1(0)$ is compact. Assume by contradiction that E is not finite dimensional. Pick an element $x_1 \in E$ with $\|x_1\| = 1$, and denote by S_1 the linear subspace generated by x_1. According to Riesz's lemma 4.28, there exists an element $x_2 \in E$ with $\|x_2\| = 1$ and $\inf_{x \in S_1} \|x - x_2\| > \frac{1}{2}$. Now let S_2 be the linear subspace generated by x_1 and x_2. Since E is not finite dimensional, S_2 is a proper subspace, and hence there exists $x_3 \in E$ with $\|x_3\| = 1$ and $\inf_{x \in S_2} \|x - x_3\| > \frac{1}{2}$. If we proceed inductively, we can construct a sequence x_n with $\|x_n\| = 1$ which satisfies $\|x_n - x_m\| > \frac{1}{2}$ for all $n \neq m$. Therefore, no convergent subsequence can be extracted from $\{x_n\}_n$, i. e. $\overline{B_1(0)}$ is not compact, a contradiction. □

4.4 *Dual spaces*

The dual space of a linear space consists of the scalar-valued linear maps on the space. Duality methods play a crucial role in many parts of analysis. In this subsection we consider real linear spaces for definiteness, but all the results hold for complex linear spaces too.

4.30 DEFINITION. A scalar-valued linear map from a linear space X to $\mathbb{R}$ is called a *linear functional*, or *linear form* on X. The space of linear functionals on X is called the *algebraic dual space* of X, and the space of continuous linear functionals on X is called the *topological dual space* of X.

In terms of notation, we denote by X^* the algebraic dual and by X' the topological dual. From now on, the topological dual will be called simply the *dual space*. In fact, $X' = \mathcal{L}(X, \mathbb{R})$. A linear functional $\varphi \in X^*$ belongs to X' if there is a constant M such that

$$|\varphi(x)| \leq M\|x\| \qquad \text{for all } x \in X,$$

and we define the dual norm of φ as the operator norm of φ, that is

$$\|\varphi\| = \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|} = \sup_{\|x\|=1} |\varphi(x)|.$$

Clearly, X^* is a linear space with the obvious structure, whereas X' is a Banach space because $\mathbb{R}$ is complete. If X is finite dimensional, then $X' = X^*$. Moreover, in this case X^* is linearly isomorphic to X. To see this, pick a basis $\{e_1, \ldots, e_n\}$ of X. The map $\omega_i : X \to \mathbb{R}$ defined by

$$\omega_i \left(\sum_{j=1}^{n} x_j e_j \right) = x_i$$

is an element of the algebraic dual space X'. The linearity of ω_i is obvious. The action of a general element φ of the dual space, $\varphi : X \to \mathbb{R}$, on a vector $x \in X$ is given by a linear combination of the components of x, since

$$\varphi \left(\sum_{i=1}^{n} x_i e_i \right) = \sum_{i=1}^{n} \varphi_i x_i,$$

where $\varphi_i = \varphi(e_i) \in \mathbb{R}$. It follows that, as a map,

$$\varphi = \sum_{i=1}^{n} \varphi_i \omega_i.$$

Thus, $\{\omega_1, \ldots, \omega_n\}$ is a basis for X', called the *dual basis* of $\{e_1, \ldots, e_n\}$, and both X' and X^* are linearly isomorphic to $\mathbb{R}^n$. The dual basis has the property that

$$\omega_i(e_j) = \delta_{ij},$$

where δ_{ij} is the *Kronecker delta function*, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Although a finite-dimensional space is linearly isomorphic with its dual space, there is no canonical way to identify the space with its dual; there are many isomorphisms, depending on the arbitrary choice of a basis. In the following chapters, we will study Hilbert spaces, and show that the topological dual space of a Hilbert space can be identified with the original space in a natural way through the inner produce. The dual of an infinite-dimensional Banach space is, in general, different from the original space.

4.31 example. Let $p \in [1, +\infty)$. We want to prove that the dual of $\ell^p(\mathbb{N})$ is (essentially) $\ell^q(\mathbb{N})$ where $1/p + 1/q = 1$. We do it in many steps.

Step 1. For all $x = (x_n) \in \ell^p(\mathbb{N})$, we have $x = \sum_{i=1}^{+\infty} x_i e_i$, where e_i denotes the usual unit vector with 0's everywhere except at the i-th component equal to 1. To prove that, we observe that $\sum_{i=1}^{n} x_i e_i \in \ell^p$ for all $n \in \mathbb{N}$, so

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\|_{\ell^p}^p = \sum_{i=n+1}^{+\infty} |x_i|^p,$$

and the last term converges to zero as $n \to +\infty$ since the series $\sum_{i=1}^{+\infty} |x_i|^p$ is convergent, as a consequence of $x \in \ell^p(\mathbb{N})$. Note that this property is *false* for $p = +\infty$ (for example, use x being a constant sequence).

Step 2. Let $f \in (\ell^p(\mathbb{N}))'$. Then there exists a $y \in \ell^q(\mathbb{N})$, $y = (\alpha_i)$ such that $f(x) = \sum_{i=1}^{+\infty} x_i \alpha_i$. To see this, let $\alpha_i = f(e_i)$. For all $x \in \ell^p(\mathbb{N})$, by continuity and linearity of f, we have

$$f(x) = \sum_{i=1}^{+\infty} x_i f(e_i) = \sum_{i=1}^{+\infty} x_i \alpha_i.$$

Consider now the case $p \in (1, +\infty)$. For $n \in \mathbb{N}$, define $x_n = \sum_{i=1}^{n} |\alpha_i|^{q/p} \mathrm{sign}(\alpha_i) e_i$. Note that $\mathrm{sign}(\alpha)\alpha = |\alpha|$. We have

$$\|x_n\|_{\ell^p} = \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/p},$$

and

$$f(x_n) = \sum_{i=1}^{n} |\alpha_i|^{q/p} \mathrm{sign}(\alpha_i) f(e_i) = \sum_{i=1}^{n} |\alpha_i|^{q/p+1} = \sum_{i=1}^{n} |\alpha_i|^q.$$

But since f is bounded, $|f(x_n)| \le \|f\|_{(\ell^p)'} \|x_n\|_{\ell^p}$, so

$$\sum_{i=1}^{n} |\alpha_i|^q \le \|f\|_{(\ell^p)'} \left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/p}.$$

Rearranging, we get

$$\left(\sum_{i=1}^{n} |\alpha_i|^q \right)^{1/q} \le \|f\|_{(\ell^p)'},$$

i.e. the sequence $y = (\alpha_i)$ belongs to ℓ^q and satisfies $\|y\|_{\ell^q} \le \|f\|_{(\ell^p)'}$. In the case $p = 1$, define $x_n = \mathrm{sign}(\alpha_n) e_n$. So $\|x_n\|_{\ell^1} \le 1$ and $f(x_n) = \alpha_n \mathrm{sign}(\alpha_n) = |\alpha_n|$, which shows

$$|\alpha_n| = |f(x_n)| \le \|f\|_{(\ell^1)'} \|x_n\|_{\ell^1} \le \|f\|_{(\ell^1)'}.$$

Hence, $y = (\alpha_i) \in \ell^\infty(\mathbb{N})$ and $\|y\|_{\ell^\infty} \le \|f\|_{(\ell^1)'}$.

Step 3. Let $y = (\alpha_i) \in \ell^q(\mathbb{N})$. We prove that y defines a functional $f_y \in (\ell^p(\mathbb{N}))'$ as follows

$$f_y(x) = \sum_{i=1}^{+\infty} x_i \alpha_i, \qquad \text{for all } x = (x_i) \in \ell^p(\mathbb{N}),$$

which has the property $\|f_y\|_{(\ell^p)'} = \|y\|_{\ell^q}$. The linearity of f_y is trivial and left as an exercise. To prove that f_y is bounded, let us observe by Hölder inequality (discrete version),

$$|f_y(x)| \le \sum_{i=1}^{+\infty} |x_i||\alpha_i| \le \left(\sum_{i=1}^{+\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{+\infty} |\alpha_i|^q\right)^{1/q} = \|x\|_{\ell^p}\|y\|_{\ell^q}.$$

The above computation also proves that $\|f\|_{(\ell^p)'} \le \|y\|_{\ell^q}$. Now, clearly $f_y(e_i) = \alpha_i$, so the same analysis in Step 1 applies and in particular $\|y\|_{\ell^q} \le \|f\|_{(\ell^p)'}$, which shows that $\|y\|_{\ell^q} = \|f\|_{(\ell^p)'}$

The example 4.31 shows that, given $p \in [1, +\infty)$ there is a map $F : \ell^q(\mathbb{N}) \to (\ell^p(\mathbb{N}))'$, with $1/q + 1/p = 1$, defined as follows: for all $y \in \ell^q(\mathbb{N})$ set $F(y) \in (\ell^p(\mathbb{N}))'$ defined by

$$F(y)(x) = \sum_{i=1}^{+\infty} x_i y_i,$$

with the following properties:

- F is one-to-one and onto (proven in example 4.31)
- F is linear (easy exercise)
- F is an isometry, i.e. $\|y\| = \|F(y)\|$ (proven in example 4.31).

So, essentially, the two normed linear spaces $\ell^q(\mathbb{N})$ and $(\ell^p(\mathbb{N}))'$ are identified.

The above identification is *false* is $p = +\infty$. It can be proven that the dual of the space c_0 of convergent sequences with limit equal to zero with the $\|\cdot\|_{\ell^\infty}$ norm is a Banach space. Moreover, the dual space of c_0 is ℓ^1. The identification (in the above sense) of the dual of ℓ^∞ goes beyond the scopes of this course.

We continue this subsection with the goal of performing a similar identification for the dual of the L^p spaces defined in subsection 3.4. Let $\Omega \subset \mathbb{R}^d$ be a measurable set. Suppose $1 \le p \le +\infty$ and $g \in L^q(\Omega)$ with $1/p + 1/q = 1$. We define $\varphi_g : L^p(\Omega) \to \mathbb{R}$ by

$$\varphi_g(f) = \int_\Omega f(x)g(x)dx \qquad \text{for every } f \in L^p(\Omega).$$

Hölder's inequality implies that φ_g is a bounded linear functional on L^p, with

$$\|\varphi_g\|_{(L^p)'} = \sup_{\|f\|_{L^p} \le 1} |\varphi_g(f)| \le \sup_{\|f\|_{L^p} \le 1} \|g\|_{L^q}\|f\|_{L^p} \le \|g\|_{L^q}.$$

Now, set $f_0 = |g|^{q-2}g$. We have

$$\varphi_g(f_0) = \int_\Omega |g(x)|^q dx = \|g\|_{L^q}^q,$$

and

$$\|f_0\|_{L^p} = \left(\int_\Omega |g(x)|^{p(q-1)} dx \right)^{1/p} = \left(\int_\Omega |g(x)|^q dx \right)^{1/p} = \|g\|_{L^q}^{q/p},$$

which implies that $f_0 \in L^p(\Omega)$ and

$$\|f_0\|_{L^p} \|g\|_{L^q} = \|g\|_{L^q}^{q/p+1} = \|g\|_{L^q}^q = |\varphi_g(f_0)|,$$

hence $\|\varphi_g\|_{(L^p)'} = \|g\|_{L^q}$.

4.32 theorem (Riesz representation Theorem for L^p spaces). *Let $\Omega \subset \mathbb{R}^d$ be a measurable set. Let $1 \le p < +\infty$. Then, for every $\varphi \in (L^p(\Omega))'$ there is a $g \in L^q(\Omega)$ with $1/p + 1/q = 1$ such that*

$$\varphi(f) = \int_\Omega f(x) g(x) dx,$$

for all $f \in L^p(\Omega)$. Moreover, $\|\varphi\|_{(L^p)'} = \|g\|_{L^q}$.

We will not give the proof of the above theorem. The identification performed before the statement of the theorem only shows that the multiplication functional $\int fgdx$ with $g \in L^q$ is a bounded (linear) functional on L^p, and that the map associating $g \in L^q$ to $\varphi \in (L^p)'$ is an isomorphism. It remains to show that such a map is onto, which goes beyond our scopes.

According to Theorem 4.32, we may identify $(L^p)'$ with L^q with $1/p + 1/q = 1$. When $p = q = 2$, we recover the result of the Riesz representation theorem on a Hilbert space, which we will prove later on. The dual of L^1 is L^∞, but the dual of L^∞ is strictly larger than L^1.

4.33 example. Consider $X = C([a, b])$. For any $g \in L^1([a, b])$, the formula

$$\varphi(f) = \int_a^b f(x) g(x) dx$$

defines a continuous linear functional φ on X. However, not all continuous functional are of the above form. For example, if $x_0 \in [a, b]$, then the evaluation of f at x_0 is a continuous linear functional. That is, if we define $\delta_{x_0} : C([a, b]) \to \mathbb{R}$ by

$$\delta_{x_0}(f) = f(x_0),$$

then δ_{x_0} is a continuous linear functional on $C([a, b])$ (easy exercise).

Since X' is a Banach space, we can form its dual space X'', called the *bidual* of X. There is not natural way to identify an element of X with an element of the dual X', but we can naturally identify an element of X with an element of the bidual X''. If $x \in X$, then we define $F_x \in X''$ by evaluation at x:

$$F_x(\varphi) = \varphi(x) \qquad \text{for every } \varphi \in X'.$$

We leave as an exercise to prove that $F_x \in X''$. In this way, we may regard X as a subspace of X''. Indeed, one can prove that the identification $X \ni x \mapsto F_x \in X''$ is isomorphic, i.e. $\|F_x\|_{X''} = \|x\|_X$ (we leave the details as an exercise). This holds for arbitrary normed spaces X, and in general such identification is not onto. Now, if *all* continuous linear functionals F on X' are of the form $F(\varphi) = \varphi(x)$ for some $x \in X$, then X and X'' essentially coincide under the identification $x \mapsto F_x$, and we say that X is *reflexive*.

If $1 < p < +\infty$, then $(L^p)'' = L^p$ and L^p is reflexive, but L^1 and L^∞ are not reflexive. Similarly to ℓ^p spaces, the situation of L^∞ is special. The dual of L^∞ is a space of measures, we omit the details and refer to [2].

4.5 *An overview of fundamental principles of functional analysis*

We provide, in this section, a brief overview of some results that are of great relevance in functional analysis, although we shall only prove some of them.

We start with a famous Theorem by Hahn and Banach, which basically says that in any linear space we can always *extend* a linear functional defined on a linear subspace in a suitable way.

A remark on the notation. In functional analysis the "action" of a functional $f \in X'$ on an element $x \in X$ is often denoted by

$$\langle f, x\rangle_{X' \times X} = \langle f, x\rangle .$$

4.34 theorem (Hahn-Banach analytic form). *Let E be a real linear space. Let $p : E \to \mathbb{R}$ be a function satisfying*

(i) $p(\lambda x) = \lambda p(x)$ for all $x \in E$ and all $\lambda > 0$,

(ii) $p(x + y) \le p(x) + p(y)$ for all $x, y \in E$.

Let $G \subset E$ be a linear subspace of E and let $g : G \to \mathbb{R}$ be a linear functional defined on G such that

$$g(x) \le p(x) \qquad \text{for all } x \in G.$$

Then, there exists a linear functional f defined on all of E that extends g, i. e. $g(x) = f(x)$ for all $x \in G$, and such that $f(x) \le p(x)$ for all $x \in E$.

Here are some important consequences of the above theorem.

4.35 corollary. *Let E be a normed space. Let $G \subset E$ be a linear subspace. If $g : G \to \mathbb{R}$ is a continuous linear functional, then there exists $f \in E'$ that extends g and such that*

$$\|f\|_{E^*} = \sup_{x \in G,\, \|x\| \le 1} |g(x)| = \|g\|_{G'}.$$

Proof. Use Theorem 4.34 with $p(x) = \|g\|_{G'}\|x\|$. □

4.36 COROLLARY. *Let E be a normed space. For every $x_0 \in E$ there exists $f_0 \in E'$ such that*

$$\|f_0\| = \|x_0\|, \qquad \text{and} \quad \langle f_0, x_0\rangle = \|x_0\|^2.$$

Proof. Use Corollary 4.35 with $G = \{tx_0 \ : \ t \in \mathbb{R}\}$ and $g(tx_0) = t\|x_0\|^2$, so that $\|g\|_{G'} = \|x_0\|$. □

In general, the element $f_0 \in E^*$ in Corollary 4.36 is not unique for a given $x_0 \in E$. It is unique in special cases, e. g. E a Hilbert space or an L^p space $p \in (1, +\infty)$, as we shall see later on. In general, for an element $x_0 \in E$ we set

$$F(x_0) = \{f_0 \in E' \ : \ \|f_0\| = \|x_0\| \text{ and } \langle f_0, x_0\rangle = \|x_0\|^2\}.$$

The multi-valued map $E \ni x_0 \mapsto F(x_0) \in \mathcal{P}(E')$ is called the *duality map* from E into E'.

4.37 COROLLARY. *Let $x_0 \in E$. If $f(x_0) = 0$ for all $f \in E'$, then $x_0 = 0$.*

Proof. Assume by contradiction that $x_0 \neq 0$. Then, by Corollary 4.36 there exists $f_0 \in E'$ with $f_0 \neq 0$ such that $\|f_0\| = \|x_0\|$ and $\langle f_0, x_0\rangle = \|x_0\|^2 \neq 0$, and this is a contradiction. □

4.38 COROLLARY. *Let E be a normed space. For every $x \in E$ we have*

$$\|x\| = \sup_{f\in E',\, \|f\|\le 1} |\langle f, x\rangle| = \max_{f\in E',\, \|f\|\le 1} |\langle f, x\rangle|.$$

Proof. The assertion is trivial if $x = 0$. If $x \neq 0$, it is clear that

$$\sup_{f\in E',\, \|f\|\le 1} |\langle f, x\rangle| \le \|x\|.$$

On the other hand, we know from Corollary 4.36 that there is some $f_0 \in E'$ such that $\|f_0\| = \|x\|$ and $\langle f_0, x\rangle = \|x\|^2$. Set $f_1(x) := \frac{f_0(x)}{\|x\|}$. We have $\|f_1\| = 1$ and $\langle f_1, x\rangle = \|x\|$. □

The above corollary is quite important, because is allows to state the following "duality principle". Given a Banach space X and its dual X', by definition of dual norm $\|f\|$ for some $f \in X'$ we have

$$\|f\| = \sup_{\|x\|=1} |\langle f, x\rangle|\,.$$

The above corollary somehow implies the opposite, namely that for all $x \in X$ we have

$$\|x\| = \sup_{\|f\|=1} |\langle f, x\rangle|\,.$$

We now state the so-called Baire's cathegory theorem, a very important but, at the same time, very abstract result, which has substantial consequences on the theory of linear operators.

Recall that a subset B of a metric space X is called *nowhere dense* if its closure has empty interior.

4.39 DEFINITION. A metric space X is called a *Baire first category space* if X can be written as the union of a countable family of nowhere dense closed sets. The space X is called a *Baire second category space* if it is not a first category space, i. e. if given a sequence closed subsets $\{F_n\}_{n\in\mathbb{N}}$ in X, if $X = \bigcup_{n=1}^{+\infty} F_n$ this implies that at least one of the F_n has a nonempty interior.

4.40 THEOREM (Baire). *Let X be a nonempty complete metric space. Then X is a Baire second category space.*

A first important consequence is the following Theorem.

4.41 THEOREM (Banach-Steinhaus, uniform boundedness principle). *Let E and F be two Banach spaces and let $\{T_i\}_{i\in I}$ be a family (not necessarily countable) of continuous linear operators from E into F. Assume that*

$$\sup_{i\in I} \|T_i(x)\| < +\infty \qquad \text{for all } x \in E. \tag{46}$$

Then,

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} < +\infty. \tag{47}$$

In other words, there exists a constant c such that

$$\|T_i(x)\| \le c\|x\| \qquad \text{for all } x \in E \text{ and for all } i \in I.$$

Proof. For every $n \ge 1$ let

$$X_n = \{x \in E \,:\, \|T_i x\| \le n \quad \text{for all } i \in I\}.$$

Clearly, X_n is closed, and from the assumption (46) we have

$$\bigcup_{n\ge 1} X_n = E.$$

From the Baire category theorem 4.40, the interior of X_{n_0} is non empty for at least one $n_0 \in \mathbb{N}$. Therefore there exists a ball $B_r(x_0) \subset X_{n_0}$ for some $x_0 \in X_{n_0}$ and some $r > 0$. This implies

$$\|T_i(x_0 + rz)\| \le n_0 \qquad \text{for all } i \in I \text{ and for all } z \in B_1(0).$$

Therefore, for all $x \in E$ with $x \neq 0$ we have

$$\frac{r}{\|x\|}\|T_i(x)\| = \left\|T_i\left(\frac{rx}{\|x\|}\right)\right\| \leq \left\|T_i\left(x_0 + \frac{rx}{\|x\|}\right)\right\| + \|T_i(x_0)\| \leq n_0 + \|T_i(x_0)\|,$$

and this proves the assertion. □

4.42 corollary. *Let E and F be two Banach spaces. Let $\{T_n\}_n$ be a sequence of continuous linear operators from E into F such that for every $x \in E$, $T_n(x)$ converges to a limit denoted by $T(x)$ as $n \to +\infty$. Then we have*

(a) $\sup_n \|T_n\|_{\mathcal{L}(E,F)} < +\infty$,

(b) $T \in \mathcal{L}(E,F)$,

(c) $\|T\|_{\mathcal{L}(E,F)} \leq \liminf_{n\to+\infty} \|T_n\|_{\mathcal{L}(E,F)}$.

Proof. (a) follows directly from Theorem 4.41, and thus there exists a constant c such that

$$\|T_n(x)\| \leq c\|x\| \qquad \text{for all } n \in \mathbb{N} \text{ and for all } n \in E.$$

Hence, for all $n \in \mathbb{N}$,

$$\|T(x)\| \leq \|(T - T_n)(x)\| + \|T_n(x)\| \leq c\|x\| + \|(T - T_n)(x)\|,$$

and taking the limit as $n \to +\infty$ we get

$$\|T(x)\| \leq c\|x\| \qquad \text{for all } x \in E.$$

Since T is clearly linear, we obtain (b).

Finally, we have

$$\|T_n(x)\| \leq \|T_n\|_{\mathcal{L}(E,F)}\|x\|.$$

By taking the lim inf on both sides, recalling that $T_n(x)$ converges to $T(x)$, we get

$$\|T(x)\| \leq \liminf_{n\to+\infty} \|T_n\|_{\mathcal{L}(E,F)}\|x\|,$$

which proves (c). □

Another important consequence of Baire's cathegory Theorem is the following result, the proof of which is non trivial and we will omit it.

4.43 theorem (Open mapping theorem). *Let E an F be two Banach spaces and let T be a continuous linear operator from E into F that is bijective (i. e. one-to-one and surjective). Then T^{-1} is also continuous from F into E.*

An important consequence of Theorem 4.43 is the following.

4.44 corollary. *Let E be a vector space provided with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that E is a Banach space with both norms and that there exists a constant $C \geq 0$ such that*

$$\|x\|_2 \leq C\|x\|_1 \qquad \textit{for all } x \in E.$$

Then the two norms are equivalent, i. e. there is a constant $c > 0$ such that

$$\|x\|_1 \leq c\|x\|_2 \qquad \textit{for all } x \in E.$$

Proof. Apply the Open Mapping Theorem with

$$E = (E, \|\cdot\|_1), \qquad F = (E, \|\cdot\|_2), \quad T = \mathbb{I}.$$

□

4.45 exercise. Let $E = L^q(\Omega)$ with $p > 1$ and with Ω a *bounded* measurable subset of $\mathbb{R}^d$. Let $q > p \geq 1$. We know there exists a constant C such that

$$\|f\|_{L^p} \leq C\|f\|_{L^q}.$$

However, the two norms are clearly not equivalent on L^p as it can be easily shown with some specific examples. Why isn't that in contradiction with the previous corollary?

The following result is an application of the open mapping theorem. It provides a useful way to show that an operator T has *closed range*, a property that is sometimes useful in the applications. The theorem states that T has closed range if one can estimate the norm of the solution x of the equation $Tx = y$ in terms of the norm of the right-hand side y.

4.46 proposition. *Let $T : X \to Y$ be a bounded linear map between Banach spaces X, Y. The following statements are equivalent:*

(a) There is a constant $c > 0$ such that

$$c\|x\| \leq \|Tx\| \qquad \textit{for all } x \in X,$$

(b) T has closed range, and the only solution of the equation $Tx = 0$ is $x = 0$.

Proof. First, suppose that T satisfies (a). Then $Tx = 0$ implies that $\|x\| = 0$, so $x = 0$. To show that $\mathrm{Ran}T$ is closed, suppose that (y_n) is a convergent sequence in $\mathrm{Ran}T$, with $y_n \to y \in Y$. Then there is a sequence (x_n) in X such that $Tx_n = y_n$. The sequence (x_n) is Cauchy, since (y_n) is Cauchy and

$$\|x_n - x_m\| \leq \frac{1}{c}\|T(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|.$$

Hence, since X is complete, we have $x_n \to x$ for some $x \in X$. Since T is bounded, we have

$$Tx = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} y_n = y,$$

so $y \in \mathrm{Ran}T$ and $\mathrm{Ran}T$ is closed.

Conversely, suppose that T satisfies (b) Since $\mathrm{Ran}T$ is closed, it is a Banach space. Since $T : X \to Y$ is one-to-one, the operator $T : X \to \mathrm{Ran}T$ os a one-to-one, onto map between Banach spaces. The open mapping theorem implies that $T^{-1} : \mathrm{Ran}T \to X$ is bounded, and hence there is a constant $C > 0$ such that

$$\|T^{-1}y\| \le C\|y\| \qquad \text{for all } y \in \mathrm{Ran}T.$$

Setting $y = Tx$, we see that $c\|x\| \le \|Tx\|$ for all $x \in X$, where $c = 1/C$. □

4.47 EXAMPLE. Consider the operator $T = \mathbb{I} + K$ on $C([0,1])$, where K is defined in (45). The range of K is the whole space $C([0,1])$ and is therefore closed. To prove this statement, we observe that $g = Tf$ if and only if

$$f(x) + \int_0^x f(y)dy = g(x).$$

Writing $F(x) = \int_0^x f(y)dy$, we have $F' = f$ and

$$F' + F = g, \qquad F(0) = 0.$$

The solution of this initial value problem is

$$F(x) = \int_0^x e^{y-x} g(y)dy.$$

Differentiating this expression with respect to x, we find that f is given by

$$f(x) = g(x) - \int_0^x e^{y-x} g(y)dy.$$

Thus, the operator $T = \mathbb{I} + K$ is invertible on $C([0,1])$ and

$$(\mathbb{I} + K)^{-1} = \mathbb{I} - L,$$

where L is the Volterra integral operator

$$Lg(x) = \int_0^x e^{y-x} g(y)dy.$$

4.48 EXAMPLE. Consider the Volterra integral operator $K : C([0,1]) \to C([0,1])$

defined in (45). Then

$$K[\cos(n\pi x)](x) = \int_0^x \cos(n\pi y)dy = \frac{\sin(n\pi x)}{n\pi}.$$

We have $\|\cos(n\pi x)\|_\infty = 1$ for every $n \in \mathbb{N}$, but $\|K[\cos(n\pi x)]\|_\infty \to 0$ as $n \to +\infty$. Thus, it is not possible to estimate $\|f\|$ in terms of $\|Kf\|$, consistent with the fact that the range of K is not closed.

4.49 theorem (Closed graph theorem). *Let E and F be two Banach spaces. Let T be a linear operator from E into F. Assume that the graph of T*

$$G(T) = \{(x, T(x)) \in E \times F \,:\, x \in E\}$$

is closed in $E \times F$. Then T is continuous.

Proof. Consider, on E, the two norms

$$\|x\|_1 = \|x\|_E + \|T(x)\|_F, \qquad \|x\|_2 = \|x\|_E$$

(the norm $\|\cdot\|_1$ is called the *graph norm*). We claim that E is a Banach space with the norm $\|\cdot\|_1$. To see this, let $\{x_n\}_n$ be a Cauchy sequence in the norm $\|\cdot\|_1$. This means that x_n is a Cauchy sequence in $\|\cdot\|_E$ and $T(x_n)$ is a Cauchy sequence in $\|\cdot\|_F$. Since both E and F are complete, $x_n \to x$ in $\|\cdot\|_E$ and $T(x_n) \to y$ for some $x \in E$ and $y \in F$. Now, since $G(T)$ is closed, the element $(x, y) \in E \times F$ belongs to $G(T)$, i. e. $y = T(x)$. This proves that x_n converges to x in the graph norm.

Now, E is also a Banach space for the norm $\|\cdot\|_2$. Hence, we can apply Corollary 4.44, which implies that the two norms are equivalent, and thus there exists a constant $c > 0$ such that $\|x\|_1 \le c\|x\|_2$. This implies $\|T(x)\|_F \le c\|x\|_E$ as desired. □

4.50 remark. The converse of the above statement is obviously true, since the graph of a any continuous map (linear or not) is closed.

4.6 *Weak topologies and weak convergences*

4.51 exercise. On a given set X one may consider two distinct topologies τ and σ. We say that τ is *weaker* than σ (or equivalently that σ is stronger than τ) if $\tau \subset \sigma$. Prove that if τ is weaker than σ then every sequence x_n converging to x in σ converges to x in τ too.

Let $(E, \|\cdot\|)$ be a normed space. Let us, for the moment, ignore the usual topology on E induced by the norm $\|\cdot\|$.

For a given family $\mathcal{F}$ of maps $f : E \to \mathbb{R}$, one can consider the *coarsest topology* τ that makes all maps $f \in \mathcal{F}$ continuous. Here $\mathbb{R}$ is considered as topological space with the usual Euclidean topology.

The topology τ is constructed as the topology *generated* by the inverse images of all open sets in $\mathbb{R}$ via all maps $f \in \mathcal{F}$. More in detail, one considers the family $\mathcal{C} = \{f^{-1}(O) : O \subset \mathbb{R} \text{ open}, f \in \mathcal{F}\}$ and take first the family $\mathcal{I}$ of the intersection of finitely many elements in $\mathcal{C}$.

Finally, one takes τ as the union of all sets in $\mathcal{I}$. Such topology is denoted by $\tau = \tau(\mathcal{C})$, and is called the *inverse limit topology* of $\mathcal{F}$. (Exercise: show $\tau(\mathcal{C})$ is a topology).

The set $\mathcal{C}$ is called a *sub-basis* for τ. More precisely, we say that a set $\mathcal{C}$ is a sub-basis for a given topology τ if τ is the coarsest (weakest) topology containing $\mathcal{C}$. Moreover, the family $\mathcal{I}$ of finite intersections in of sets in $\mathcal{C}$ is a *basis* for the topology τ. This means that every open set in $\tau(\mathcal{C})$ can be written as union of sets in $\mathcal{I}$ (exercise!).

Finally, for a given point $x \in E$, the family

$$\begin{aligned}\mathcal{U}_x = \{&f_1^{-1}(O_1) \cap \ldots \cap f_k^{-1}(O_k) : O_1, \ldots, O_k \\ &\text{open in } \mathbb{R}, f_1, \ldots, f_k \in \mathcal{F}, \text{ with } f_j(x) \in O_j \text{ for all } j = 1, \ldots, k\}\end{aligned}$$

is a *basis of neighborhoods* for x in the topology τ, which means that every open neighborhood of x in τ contains an open set of the family $\mathcal{U}_x$.

4.52 exercise. With the notation above, prove that a sequence $\{x_n\}_n \subset E$ converges to $x \in E$ in $\tau(\mathcal{C})$ if and only if $f(x_n) \to f(x)$ for all $f \in \mathcal{F}$.

Solution: If $x_n \to x$ in τ, then $f(x_n) \to f(x)$ for all $f \in \mathcal{F}$ because all $f \in \mathcal{F}$ are continuous in the topology τ (the image of an open set O via f is open in $\tau(\mathcal{C})$!). Vice versa, assume $f(x_n) \to f(x)$ for all $f \in \mathcal{F}$. Let $U_x \in \mathcal{U}_x$ be a basic neighborhood of x, in particular, $U_x = f_1^{-1}(O_1) \cap \ldots \cap f_k^{-1}(O^k)$ for some $f_1, \ldots, f_k \in \mathcal{F}$ and some open sets $O_1, \ldots, O_k \subset \mathbb{R}$ with $f_i(x) \in O_i$ for all $i = 1, \ldots, k$. By assumption, $f_i(x_n) \to f_i(x)$, hence there exist integers $N_i \in \mathbb{N}$ such that $f_i(x_n) \in O_i$ for all $n \geq N_i$, for all $i = 1, \ldots, k$. Set $N := \max\{N_1, \ldots, N_k\}$. Then, for all $n \geq N$ one has $f_i(x_n) \in O_i$ for all $i = 1, \ldots, k$, i. e. $x_n \in f_1^{-1}(O_1) \cap \ldots \cap f_k^{-1}(O_k)$, i. e. $x_n \in U_x$. This implies that $x_n \to x$ in τ.

4.53 exercise. With the notation above, let Z be a topological space and let $\psi : Z \to E$. Then ψ is continuous if and only if $f \circ \psi$ is continuous from Z into $\mathbb{R}$ for every $f \in \mathcal{F}$.

Solution: If ψ is continuous then $f \circ \psi$ is also continuous for all $f \in \mathcal{F}$ (all the $f \in \mathcal{F}$ are continuous in τ, and the composition of continuous functions is continuous). Vice versa, assume $f \circ \psi$ is continuous for every $f \in \mathcal{F}$. We have to prove that $\psi^{-1}(U)$ is open for all sub-basic open sets $U \subset E$ in the τ topology. But we know that sub basic open sets in τ are all of the form $U = f^{-1}(O)$ for some $f \in \mathcal{F}$ and some $O \in \mathbb{R}$ open. Hence, $\psi^{-1}(U) = \psi^{-1}(f^{-1}(O)) = (f \circ \psi)^{-1}(O)$, and the latter is an open set by the continuity of $f \circ \psi$.

Let E be a normed space and let $f \in E'$. We denote by $\varphi_f : E \to \mathbb{R}$ the linear functional $\varphi_f(x) = \langle f, x \rangle$. As f runs through E^* we obtain a collection $\{\phi_f\}_{f \in E'}$ of maps from E into $\mathbb{R}$. We now ignore the usual topology induced by $\| \cdot \|$ on E and define a new topology on the set E as follows.

4.54 definition. The *weak topology* $\sigma(E, E')$ on E is the inverse limit topology of the family of maps $\{\phi_f\}_{f\in E'}$.

4.55 remark. Note that all $f \in E^*$ are *continuous* functionals on $(E, \|\cdot\|)$. Since $\sigma(E,E')$ is the coarsest topology that makes all $f \in E'$ continuous, we deduce that *the weak topology $\sigma(E,E')$ on E is weaker than the usual topology induced on E by the norm $\|\cdot\|$*, which we will from now on refer to as the *strong* topology on E.

4.56 proposition. *Let $x_0 \in E$. Given $\epsilon > 0$ and a finite set $\{f_1, f_2, \dots, f_k\} \subset E'$, consider*

$$V = V(f_1,\dots,f_k;\epsilon) = \{x \in E \,:\, |\langle f_i, x - x_0\rangle| < \epsilon, \text{ for all } i = 1,\dots,k\}.$$

Then V is a neighborhood of x_0 for the topology $\sigma(E,E')$. Moreover, we obtain a basis of neighborhoods of x_0 by varying ϵ, k, and the f_i's in E'.

Proof. From the discussion above we already know that the sets V in the statement are a basis of neighborhoods for x_0 if the statement $|\langle f_i, x - x_0\rangle| < \epsilon$ is replaced by $f_i \in O_i$ for some O_i open neighborhood of $f_i(x_0)$. The statement follows by recalling that the open sets of $\mathbb{R}$ are characterised as the sets O such that for all $y \in O$ there exists an open interval $(y-\epsilon, y+\epsilon) \subset O$ for some $\epsilon > 0$. And this proves the assertion. $\square$

Let $\{x_n\}_n$ be a sequence on E. If x_n converges to $x \in E$ in the $\sigma(E,E')$ topology we shall use the notation

$$x_n \rightharpoonup x.$$

We shall sometimes say $x_n \rightharpoonup x$ *weakly in* $\sigma(E,E')$. The convergence of x_n to x in the usual topology will be sometimes emphasised by saying $x_n \to x$ *strongly*, meaning $\|x_n - x\| \to 0$.

4.57 proposition. *Let $\{x_n\}_n \subset E$ be a sequence in E. Then*

(i) $x_n \rightharpoonup x$ *weakly in $\sigma(E,E')$ is and only if $\langle f, x_n\rangle \to \langle f, x\rangle$ for all $f \in E'$.*

(ii) If $x_n \to x$ strongly, then $x_n \rightharpoonup x$ weakly in $\sigma(E,E')$.

(iii) If $x_n \rightharpoonup x$ weakly in $\sigma(E,E')$, then $\{\|x_n\|\}_n$ is bounded and

$$\|x\| \le \liminf_{n\to+\infty} \|x_n\|.$$

(iv) If $x_n \rightharpoonup x$ weakly in $\sigma(E,E')$ and if $f_n \to f$ strongly in E' (i. e. $\|f_n - f\|_{E'} \to 0$), then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Proof. (i) is a consequence of the Exercise 4.52 and the definition of weak topology $\sigma(E,E')$ on E.

(ii) is a consequence of (i), since

$$|\langle f, x_n\rangle - \langle f, x\rangle| \le \|f\|_{E'}\|x_n - x\|.$$

Alternatively, it is a consequence of the fact that the weak topology is weaker than the norm topology.

(iii) follows from Theorem 4.41. Indeed, for every $n \in \mathbb{N}$ define the map $T_n : E' \to \mathbb{R}$ as $T_n(f) = \langle f, x_n \rangle$. Since x_n converges weakly to x, for every $f \in E'$ the real sequence $\langle f, x_n \rangle$ is convergent, and hence bounded. Therefore, for all $f \in E'$ the family $T_n(f)$ is uniformly bounded in E' with respect to $n \in \mathbb{N}$. Hence, from the Banach-Steinhaus theorem 4.41 we have $\sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{L}(E',\mathbb{R})} < +\infty$ and there exists a constant $c \in \mathbb{R}$ such that

$$|T_n(f)| = |\langle f, x_n \rangle| \leq c\|f\|_{E'}, \qquad \text{for all } n \in \mathbb{N}.$$

Hence, Corollary 4.38 implies

$$\|x_n\| = \sup_{f\in E',\ \|f\|_{E'}\leq 1} |\langle f, x_n \rangle| \leq c,$$

which proves that $\|x_n\|$ is a bounded sequence. Now, taking the $\liminf_{n\to+\infty}$ in the inequality $|\langle f, x_n \rangle| \leq \|f\|_{E'}\|x_n\|$ we obtain

$$|\langle f, x \rangle| \leq \|f\|_{E'} \liminf_{n\to+\infty} \|x_n\|,$$

which implies, once again by Corollary 4.38,

$$\|x\| = \sup_{\|f\|\leq 1} |\langle f, x \rangle \leq \liminf_{n\to+\infty} \|x_n\|.$$

(iv) follows from the inequality

$$|\langle f_n, x_n \rangle - \langle f, x \rangle| \leq |\langle f_n - f, x_n \rangle| + |\langle f, x_n - x \rangle| \leq \|f_n - f\|\|x_n\| + |\langle f, x_n - x \rangle|.$$

Now, due to (iii) $\|x_n\|$ is uniformly bounded and the first term in the r.h.s. above goes to zero. The second term goes to zero because of (i). □

4.58 proposition. *When E is finite-dimensional, the weak topology $\sigma(E, E')$ and the usual topology are the same. In particular, a sequence $\{x_n\}_n$ converges weakly if and only if it converges strongly.*

Proof. Since the weak topology is always weaker than the strong topology, it suffices to check that every strongly open set is weakly open. Let $x_0 \in E$ and let U be a neighborhood of x_0 in the strong topology. We have to find a neighborhood V of x_0 in the weak topology $\sigma(E, E')$ such that $V \subset U$. In other words, we have to find $f_1, \ldots, f_k \in E'$ and $\epsilon > 0$ such that

$$V = \{x \in E : |\langle f_i, x - x_0 \rangle| < \epsilon, \text{ for all } i = 1, \ldots, k\} \subset U.$$

Fix $r > 0$ such that $B_r(x_0) \subset U$. Pick a basis $e_1, e_2, \ldots, e_k \in E$ such that $\|e_i\| = 1$ for all i. Hence, every $x \in E$ can be written as $x = \sum_{i=1}^k x_i e_i$, and the maps $x \mapsto x_i$ are continuous linear functionals on E denoted by f_i. Choosing those functionals for the neighborhood V we have, for all $x \in V$,

$$\|x - x_0\| \leq \sum_{i=1}^{k} |\langle f_i, x - x_0 \rangle| < k\epsilon.$$

Choosing $\epsilon = r/k$, we obtain $V \subset B_r(x_0) \subset U$ as desired. $\square$

Open (resp. closed) sets in the weak topology $\sigma(E, E')$ are always open (resp. closed) in the strong topology. In any infinite dimensional space the weak topology is strictly coarser than the strong topology: i. e., there exist open (resp. closed) sets in the strong topology that are not open (resp. closed) in the weak topology. Here are two examples.

4.59 example. The unit sphere $S = \{x \in E : \|x\| = 1\}$, with E infinite dimensional, is *never* closed in the weak topology $\sigma(E, E')$. More precisely, we have

$$\overline{S}^{\sigma(E,E')} = \overline{B_1(0)} = \{x \in E : \|x\| \leq 1\},$$

where $\overline{S}^{\sigma(E,E')}$ denotes the closure of S in the topology of $\sigma(E, E')$. First, let us check that every $x_0 \in E$ with $\|x_0\| < 1$ belongs to $\overline{S}^{\sigma(E,E')}$. Indeed, let V be a neighborhood of x_0 in $\sigma(E, E')$. We have to prove that $V \cap S \neq \emptyset$. In view of Proposition 4.56, we may always assume that V has the form

$$V = \{x \in E : |\langle f_i, x - x_0 \rangle| < \epsilon \text{ for all } i = 1, \dots, k\},$$

with $\epsilon > 0$ and $f_1, \dots, f_k \in E'$.

Now, we claim that there exists $y_0 \in E$, $y_0 \neq 0$, such that $\langle f_i, y_0 \rangle = 0$ for all $i = 1, \dots, k$. Assume by contradiction that the claim is false. We define the following linear map $\varphi : E \to \mathbb{R}^k$,

$$\varphi(x) = (\langle f_1, x \rangle, \dots, \langle f_k, x \rangle) \in \mathbb{R}^k.$$

If the claim is false, then for all $x \in E$ there exists $i \in \{1, \dots, k\}$ such that $f_i(x) \neq 0$, which means that for all $x \in E$ the vector $\varphi(x) \in \mathbb{R}^k$ is always non zero. Hence, φ is a linear isomorphism from E onto $\varphi(E)$, and hence $\dim(E) = \dim(\varphi(E)) \leq k$, which contradicts the assumption that E has infinite dimension. This proves the claim.

Now, the function $[0, +\infty) \ni t \mapsto g(t) := \|x_0 + ty_0\|$ is continuous on $[0, +\infty)$. Moreover, $g(0) < 1$, and $\lim_{t \to +\infty} g(t) = +\infty$. Hence, there exists $t_0 > 0$ such that $\|x_0 + ty_0\| = 1$, i. e. $x_0 + t_0 y_0 \in S$. Now, for all $i \in 1, \dots, k$ we have $\langle f_i, (x_0 + t_0 y_0) - x_0 \rangle = t_0 \langle f_i, y_0 \rangle = 0 < \epsilon$, clearly $x_0 + t_0 y_0 \in V$. Therefore, $x_0 + t_0 y_0 \in V \cap S$.

The above argument proves that $S \subset \overline{B_1(0)} \subset \overline{S}^{\sigma(E,E')}$. To prove $\overline{B_1(0)} \supset \overline{S}^{\sigma(E,E^*)}$ it suffices to prove that $\overline{B_1(0)}$ is closed in the weak topology. Now, for a given $x \in E$ with $\|x\| \leq 1$ we have by Corollary 4.38 that $\sup_{\|f\| \leq 1} |\langle f, x \rangle| \leq 1$, hence $x \in \{x \in E : |\langle f, x\rangle| \leq 1\}$ for all $f \in E'$ with $\|f\|_{E'} \leq 1$, i. e.

$$\overline{B_1(0)} \subset \bigcap_{f \in E',\ \|f\| \leq 1} \{x \in E : |\langle f, x\rangle| \leq 1\}.$$

Moreover, again by Corollary 4.38 every element $x \in \cap_{f \in E^*,\ \|f\| \leq 1}\{x \in E : |\langle f, x\rangle| \leq 1\}$ satisfies

$$\|x\| = \sup_{\|f\| \leq 1} |\langle f, x\rangle| \leq 1,$$

and this proves that actually

$$\overline{B_1(0)} = \bigcap_{f \in E^*,\ \|f\| \leq 1} \{x \in E : |\langle f, x\rangle| \leq 1\}.$$

Since the r.h.s. is the intersection of closed sets in the weak topology $\sigma(E, E')$, this implies that $\overline{B_1(0)}$ is weakly closed.

4.60 Example. The open unit ball

$$U = \{x \in E : \|x\| < 1\},$$

with E infinite dimensional, is *never* open in the weak topology $\sigma(E, E')$. Suppose by contradiction that U is weakly open. Then its complement $U^c = \{x \in E : \|x\| \geq 1\}$ is weakly closed. It follows that $\overline{B_1(0)} \cap U^c = S$ is weakly closed, which contradicts the example 4.59.

One can prove that the weak topology is never metrizable in infinite dimensions, i. e. there is no metric on E that induces the weak topology $\sigma(E, E')$. The proof is postponed.

Keep in mind that, in general, if two topological spaces have the same convergent sequences this does not automatically imply that they have the same topologies. Indeed, if two *metric spaces* have the same convergent sequences then they have the identical topologies. But in general, the set of convergent sequences is not enough to characterise a topology if this is not induced by a distance. This discussion implies in particular that, in principle, there might be examples of infinite dimensional spaces E for which every weakly converging sequence also converges strongly, whereas of course the two topologies (strong and weak) are distinct. Such examples are quite rare and pathological.

We now turn our attention to the dual space E'. As we know, E' is a normed space with the operator norm

$$\|f\|_{E'} = \sup_{\|x\| \leq 1} |f(x)|.$$

Hence, one could consider the dual space of E', i. e. the space of all continuous linear functionals defined on E', given by the bidual E''. As dual of the space

E^*, the space E'' is naturally equipped with the norm

$$\|\xi\|_{E''} = \sup_{f \in E',\, \|f\|_{E'} \le 1} |\langle \xi, f \rangle|, \qquad \xi \in E''.$$

Recall the *canonical injection* $J : E \to E''$ defined as follows. Given $x \in E$, the map $E' \ni f \mapsto \langle f, x \rangle$ is a continuous linear functional on E'. This is due to the inequality

$$|\langle f, x \rangle| \le \|x\|_E \|f\|_{E'}.$$

Thus, such map is an element of E''. We denote such element as $J(x)$. By definition, we have

$$\langle J(x), f \rangle_{E'', E'} = \langle f, x \rangle_{E', E}, \qquad \text{for all } f \in E'.$$

It is clear that J is linear and that J is an *isometry*, that is, $\|J(x)\|_{E''} = \|x\|_E$. Indeed, we have

$$\|J(x)\|_{E''} = \sup_{\|f\|_{E'} \le 1} |\langle J(x), f \rangle| = \sup_{\|f\|_{E'} \le 1} |\langle f, x \rangle| = \|x\|_E,$$

and the last step is due to Corollary 4.38.

So far, we have two topologies on E':

(a) the usual strong topology associated to $\| \cdot \|_{'}$,

(b) the weak topology $\sigma(E', E'')$.

We are now going to define a *third topology* on E', called weak* topology and denoted by $\sigma(E', E)$. For every $x \in E$ consider the linear functional $\varphi_x : E' \to \mathbb{R}$ defined by $E' \ni f \mapsto \varphi_x(f) = \langle f, x \rangle$. As x runs through E we obtain a collection $\{\varphi_x\}_{x \in E}$ of maps from E' into $\mathbb{R}$.

4.61 definition. The *weak* topology* $\sigma(E', E)$ on E' is the inverse limit topology of the family of maps $\{\varphi_x\}_{x \in E}$.

Notice that φ_x is just another notation for $J(x)$ above. Hence, for all $x \in E$, the linear functional $\varphi_x : E' \to \mathbb{R}$ is continuous (as a function from the normed space $(E^*, \| \cdot \|_{E'})$ to $\mathbb{R}$), and therefore $\varphi_x \in E''$. This fact implies that the weak* topology $\sigma(E', E)$ on E' is coarser than the weak topology $\sigma(E', E'')$ on E', i. e. $\sigma(E', E)$ has fewer open sets (resp. closed sets) than $\sigma(E', E'')$, which in turn has fewer open sets than the strong topology induced on E' by the operator norm $\| \cdot \|_{E'}$.

One may probably wonder why there is such an interest into defining weaker and weaker topologies. The reason is the following: *a coarser topology has more compact sets*, since there are *less open covers* to test the compactness condition.

We now state some general properties of the weak* topology without proofs.

4.62 proposition. *Let $f_0 \in E'$. Given a finite set $\{x_1, \dots, x_k\} \subset E$ and $\epsilon > 0$, consider*

$$V = V(x_1, \dots, x_k; \epsilon) = \{f \in E' \,:\, |\langle f - f_0, x_i\rangle| < \epsilon, \text{ for all } i = 1, \dots, k\}.$$

Then V is a neighborhood of f_0 for the topology $\sigma(E', E)$. Moreover, we obtain a basis of neighborhoods of f_0 for $\sigma(E', E)$ by varying ϵ, k, and the x_i's in E.

Let $\{f_n\}_n$ be a sequence on E'. If f_n converges to $f \in E'$ in the $\sigma(E', E)$ topology we shall use the notation

$$f_n \overset{*}{\rightharpoonup} f.$$

We shall sometimes say $f_n \overset{*}{\rightharpoonup} f$ *weakly* in* $\sigma(E', E)$. The convergence of f_n to f in the weak topology $\sigma(E', E'')$ will be denoted by $f_n \rightharpoonup f$ in $\sigma(E', E'')$. The convergence of f_n to f in the strong topology of E' will be sometimes emphasised by saying $f_n \to f$ *strongly*, meaning $\|f_n - f\|_{E'} \to 0$.

By mimicking the proof of Proposition 4.57, one can prove (we won't) the following statements for a general sequence $f_n \in E'$:

(i) $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E', E)$ if and only if $\langle f_n, x\rangle \to \langle f, x\rangle$ for all $x \in E$.

(ii) If $f_n \to f$ strongly, then $f_n \rightharpoonup f$ in $\sigma(E', E'')$. If $f_n \rightharpoonup f$ in $\sigma(E', E'')$, then $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E', E)$.

(iii) If $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E', E)$ then $\{\|f_n\|\}_n$ is bounded and $\|f\| \le \liminf_{n\to+\infty} \|f_n\|$.

(iv) If $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E', E)$ and if $x_n \to x$ strongly in E, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

When $\dim(E) < +\infty$ the three topologies (strong, weak, weak*) on E' coincide. Indeed, the canonical injection $J : E \to E''$ is in this case surjective (since $\dim(E) = \dim(E'')$ and therefore $\sigma(E', E) = \sigma(E', E'')$.

We are now ready to state one of the main results of this part. As we observed above, weakening a topology implies having more compact sets. As seen in Theorem 4.29, one of the main points with the strong topology on a normed space of infinite dimension is that the closed unit ball is not a compact set. Such a situation changes drastically when passing from the strong topology to weak topologies. The next result is a first big step toward this direction.

4.63 theorem (Banach - Alaoglu - Bourbaki). *The closed unit ball*

$$B_{E'} = \{f \in E' \,:\, \|f\|_{E'} \le 1\}$$

is compact in the weak topology $\sigma(E', E)$.*

The next result describes a basic property of reflexive spaces.

4.64 theorem (Kakutani). *Let E be a Banach space. Then E is reflexive if and only if*

$$B_E = \{x \in E \,:\, \|x\| \leq 1\}$$

is compact in the weak topology $\sigma(E, E')$.

Other relevant properties are stated below.

4.65 proposition. *Assume that E is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of E. Then M is reflexive.*

4.66 proposition. *A Banach space E is reflexive if and only if its dual space E' is reflexive.*

An important role is played, at this stage, by *separability*.

4.67 theorem. *Let E be a Banach space such that E' is separable. Then, E is separable.*

4.68 corollary. *Let E be a Banach space. Then, E is reflexive and separable if and only if E' is reflexive and separable.*

We conclude with a result which is probably the most important one of this section.

4.69 theorem (Weak compactness on reflexive spaces). *Assume that E is a reflexive Banach space and let $\{x_n\}_n$ be a bounded sequence in E. Then there exists a subsequence x_{n_k} that converges in the weak topology $\sigma(E, E')$.*

4.7 *Weak convergences in ℓ^p and L^p spaces*

4.70 example. Let $p \in (1, +\infty)$ and $X = \ell^p(\mathbb{N})$. By definition, a sequence $(x_n) \in X$ converges weakly to x if $\varphi(x_n) \to \varphi(x)$ as $n \to +\infty$ for all $\varphi \in (\ell^p(\mathbb{N}))'$. Example 4.31 shows that this is equivalent to requiring

$$\sum_{k=1}^{+\infty} x_{n,k} y_k \to \sum_{k=1}^{+\infty} x_k y_k \qquad \text{as } n \to +\infty$$

for all $y = (y_k) \in \ell^q(\mathbb{N})$ with $1/p + 1/q = 1$. We show with the following example that weak convergence in general does not imply strong convergence. Consider the sequence (x_n) in $\ell^p(\mathbb{N})$ defined by $x_n = (x_{n,k})_k$ with $x_{n,k} = \delta_{n,k}$, $\delta_{n,k}$ being the usual Kronecker delta. We show that x_n converges weakly to zero in $\ell^p(\mathbb{N})$. To see that, let $y = (y_k)$ be an element of $\ell^q(\mathbb{N})$ with $1/q + 1/p = 1$. Compute

$$\sum_{k=1}^{+\infty} x_{n,k} y_k = \sum_{k=1}^{+\infty} \delta_{n,k} y_k = y_n.$$

Since $y \in \ell^q(\mathbb{N})$, the series $\sum_{k=1}^{+\infty} |y_n|^q$ converges, which implies that $y_n \to 0$ as $n \to +\infty$, and the assertion is proven. However, x_n does not converge strongly to zero. Indeed,

$$\|x_n - 0\|_{\ell^p}^p = \sum_{k=1}^{+\infty} |x_{n,k}|^p = 1 \nrightarrow 0.$$

Such an argument clearly does not work if $p = 1$. In this case, the dual space is identified with $\ell^\infty(\mathbb{N})$, and therefore it is no longer true that $y_n \to 0$ as $n \to +\infty$. In fact one can prove that $\ell^1(\mathbb{N})$ has the so-called *Schur property*, namely that every weakly convergent sequence is also strongly convergent, we omit the details.

Let us now consider the case $p = +\infty$. In this case, we cannot easily identify the dual space of $X = \ell^\infty(\mathbb{N})$, hence the weak convergence is difficult to state. But since X is the dual of $\ell^1(\mathbb{N})$, we can easily define weak-$*$ convergence as follows: a sequence $(x_n) \in \ell^\infty(\mathbb{N})$ converges weakly-$*$ to x if and only if

$$\sum_{k=1}^{+\infty} x_{n,k} y_k \to \sum_{k=1}^{+\infty} x_k y_k \qquad \text{as } n \to +\infty$$

for all $y = (y_k) \in \ell^1(\mathbb{N})$. The above example $x_{n,k} = \delta_{n,k}$ works also in this case to show that x_n converges to zero in the weak-$*$ sense but not strongly.

Let us now turn to weak convergence in L^p spaces. According to Theorem 4.32, weak convergence can be characterized in terms of convergence of multiplications under integrals. This permits us to reformulate the notion of weak convergence in L^p spaces as follows. Here we shall always think of L^p as $L^p(\Omega)$ for some measurable set $\Omega \subset \mathbb{R}^d$.

4.71 definition. Suppose that $1 \leq p < +\infty$. A sequence (f_n) *converges weakly* to f in L^p, written $f_n \rightharpoonup f$, if

$$\lim_{n \to +\infty} \int f_n g dx = \int f g dx \qquad \text{for every } g \in L^q,$$

where q is the Hölder conjugate of p, $1/p + 1/q = 1$. When $p = +\infty$ and $q = 1$, the condition above corresponds to weak-$*$ convergence of f_n to f in L^∞.

As in the case of ℓ^p spaces, weak L^p convergence does not imply strong L^p-convergence, i.e. convergence in the L^p norm. The following example illustrates three typical ways in which a weakly convergent sequence of functions can fail to be strongly convergent.

4.72 example. Let $g \in L^p(\mathbb{R})$ be a fixed nonzero function, where $1 < p < +\infty$. For each of the following three sequences, we have $f_n \rightharpoonup 0$ weakly as $n \to +\infty$ but not $f_n \to 0$ strongly, in $L^p(\mathbb{R})$.

(a) $f_n(x) = g(x) \sin nx$ (oscillation). Consider the case $g = \mathbf{1}_{[0,\pi]}$. For every polynomial p we have

$$\int_{\mathbb{R}} f_n p dx = \int_0^{\pi} \sin nx p(x) dx = \frac{1}{n}\left[p(0) - p(\pi)\cos n\pi + \int_0^{\pi} p'(x)\cos nx dx \right],$$

hence $\int_{\mathbb{R}} f_n p dx \to 0$ as $n \to +\infty$. We know that polynomial are dense in $C([0,\pi])$. Therefore, for a general $f \in L^q(\mathbb{R})$ with $1/q + 1/p = 1$, and an arbitrary $\epsilon > 0$, let p be a polynomial on $[0,\pi]$ such that $\|f - p\|_{L^\infty([0,\pi])} < \epsilon$. Consider

$$\begin{aligned}\left|\int_{\mathbb{R}} f_n f dx\right| &\le \left|\int_{\mathbb{R}} f_n p dx\right| + \left|\int_0^{\pi} f_n (f-p) dx\right| \\ &\le \left|\int_{\mathbb{R}} f_n p dx\right| + \|f_n\|_{L^1([0,\pi])}\|f-p\|_{L^\infty([0,\pi])}.\end{aligned}$$

Now, the last term above can be controlled by

$$\|f_n\|_{L^1([0,\pi])}\|f-p\|_{L^\infty([0,\pi])} \le \epsilon \int_0^{\pi} |\sin nx| dx \le \epsilon,$$

and then we can send $n \to +\infty$ and get

$$\lim_{n\to+\infty}\left|\int_{\mathbb{R}} f_n f dx\right| \le \epsilon,$$

in view of the previous case of polynomials. Since ϵ is arbitrary, we have proven that $f_n \rightharpoonup 0$ weakly in L^p. On the other hand, f_n does not converge strongly to zero in L^p, since $\int_0^{\pi} |\sin(nx)|^p dx$ can be easily proven not to converge to zero as $n \to +\infty$ (exercise).

- $f_n(x) = n^{1/p} g(nx)$ (concentration). Once again, for simplicity let us consider the case $g = \mathbf{1}_{[0,\pi]}$. For all $f \in L^q(\mathbb{R})$ with q conjugate exponent of p, Hölder's inequality implies

$$\begin{aligned}\left|\int_{\mathbb{R}} f_n(x) f(x) dx\right| &= n^{1/p}\int_0^{1/n} |f(x)| dx \\ &\le n^{1/p}\left(\int_0^{1/n} dx\right)^{1/p}\left(\int_0^{1/n} |f(x)|^q dx\right)^{1/q} \\ &= n^{1/p}\frac{1}{n^{1/p}}\left(\int_0^{1/n} |f(x)|^q dx\right)^{1/q} = \left(\int_0^{1/n} |f(x)|^q dx\right)^{1/q}.\end{aligned}$$

Now, the sequence of functions $h_n = |f|^q \mathbf{1}_{[0,1/n]}$ satisfy $0 \leq h_n \leq |f|^q$, and the latter is a summable function. Therefore, Lebesgue's dominated convergence theorem easily implies $\left(\int_0^{1/n} |f(x)|^q dx\right)^{1/q} \to 0$ as $n \to +\infty$. This shows that $f_n \rightharpoonup 0$ weakly in $L^p(\mathbb{R})$. On the other hand,

$$\|f_n\|^p_{L^p(\mathbb{R})} = \int_0^{1/n} |n^{1/p}|^p dx = 1 \qquad \text{for all } n \to +\infty,$$

which implies f_n does not converge to zero in L^p.

(c) $f_n(x) = g(x-n)$ (escape to infinity). Using the example $g(x) = \mathbf{1}_{[0,1]}$, it is immediately seen that f_n does not converge to zero in L^p. On the other hand, for all $f \in L^q(\mathbb{R})$,

$$\left|\int f_n f dx\right| = \left|\int_n^{n+1} f dx\right| \leq \left(\int_n^{n+1} |f(x)|^q dx\right)^{1/q},$$

and the last term converges to zero similarly to case (b) above.

4.8 *Exercises*

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} x - \frac{1}{2n} & x \geq 1/n \\ \frac{nx^2}{2} & -1/n \leq x \leq 1/n \\ -x - \frac{1}{2n} & x \leq -1/n. \end{cases}$$

Prove that f_n is continuously differentiable for all $n \in \mathbb{N}$. Find (if it exists) f the uniform limit as $n \to +\infty$ of f_n on $\mathbb{R}$. Is f differentiable?

2. Is the space $C([0,1])$ complete with respect to the L^p norm for $p \in [1, +\infty)$? Justify your answer.

3. Prove that the set

$$\{f \in C([0,1]) \ : \ f(0) = 0\}$$

is a closed linear subspace of $C([0,1])$.

4. Consider the operator $K : C([0,1]) \to C([0,1])$ defined by

$$Kf(x) = \int_0^1 \sin(\pi(x-y)) f(y) dy.$$

   (a) Prove that K is a bounded linear operator.

   (b) Find the range of K.

5. Let $f(x) = |x|$ be defined on $x \in [-1, 1]$.
   - Prove that the function f is an element of the Sobolev space $W^{1,p}$ for all $p \in [1, +\infty]$ (Hint: show that the $W^{1,p}$ norms of f are finite for all p).
   - For $n \in \mathbb{N}$ consider the sequence

   $$f_n(x) = \begin{cases} -x - \frac{1}{2n} & \text{if } -1 \le x < -1/n \\ n\frac{x^2}{2} & \text{if } -1/n \le x \le 1/n \\ x - \frac{1}{2n} & \text{if } 1/n < x \le 1. \end{cases}$$

   Show that $f_n \to f$ as $n \to +\infty$ in $W^{1,p}$ for all $p \in [1, +\infty)$ but not for $p = +\infty$.

6. Consider the operator $T : L^1([0,1]) \to C([0,1])$

   $$(Tf)(x) = \int_0^x t f(t) dt.$$

   - Prove that T is a linear operator.
   - Prove that T is a bounded operator.

7. Suppose that $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function. Prove that the integral operator $K : C([0,1]) \to C([0,1])$ defined by

   $$Kf(x) = \int_0^1 k(x,y) f(y) dy$$

   is compact.

8. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

   $$(Tx)_n = \arctan(n) x_n \qquad n \in \mathbb{N}.$$

   Show that T is a bounded linear operator and compute the operator norm $\|T\|$.

9. Let $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ defined by

   $$(Tx)_n = \frac{n^2}{1+n^2}(x_n + x_{n+1}) \qquad n \in \mathbb{N}.$$

   Show that T is a bounded linear operator and compute the operator norm $\|T\|$.

10. Let $g : \mathbb{R}^d \to [0, +\infty)$ be a measurable nonnegative function, and let $p \in (1, +\infty)$. Consider the operator

    $$f \mapsto Tf(x) = g(x) f(x).$$

Find a condition on g such that the above operator T is a linear and bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.

11. Consider the operator $A : C^1((0,1)) \to C((0,1))$ defined by

$$(Af)(x) = \frac{d}{dx}f(x).$$

Show that A is linear but not bounded.

12. Let $\ell_c(\mathbb{N})$ be the space of all real-valued sequences of the form $x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$, whose terms vanish from some point onwards.

(a) Prove that $\ell_c(\mathbb{N})$ is an infinite dimensional linear subspace of $\ell^p(\mathbb{N})$ for all $p \in [1, +\infty]$.

(b) Prove that $\ell_c(\mathbb{N})$ is not closed in $\ell^p(\mathbb{N})$ for all $p \in [1, +\infty]$.

(c) Prove that $\ell_c(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$ for all $p \in [1, +\infty)$.

(d) Prove that the closure of $\ell_c(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$ is the space of all sequences that converge to zero.

13. Let $T : L^2([0,\pi]) \to L^2([0,\pi])$ be defined as

$$(Tf)(x) = \int_0^\pi \cos(x+2y)f(y)dy.$$

Find the kernel and the range of T.

14. Let $T : L^1(\mathbb{R}) \to \mathbb{R}$ be defined as

$$T(f) = \int_{\mathbb{R}} \sin x f(x)dx.$$

Show that T is a linear and continuous functional on L^1 and compute its norm.

15. Let $x_0 \in [0,1]$. Let $T : C([0,1]) \to \mathbb{R}$ be defined as

$$T(f) = f(x_0).$$

Prove that T is a linear and bounded functional and compute its norm.

16. Let (x_n) be the sequence in ℓ^2 defined by

$$x_{n,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

Prove that x_n converges to zero weakly in ℓ^2. Is (x_n) converging to zero strongly?

17. $f_n : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_n(x) = \sqrt{n}\mathbf{1}_{[0,1/n]}.$$

Prove that f_n converges weakly to zero in $L^2(\mathbb{R})$.

18. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = n\mathbf{1}_{[0,1/n]}(x)$.

    (a) Prove that f_n is uniformly bounded in $L^1(\mathbb{R})$.

    (b) Is it possible to extract a subsequence of f_n which converges weakly in L^1?

4.9 *Envisaged outcomes*

At the end of this chapter, the student should

- Have a clear picture of the main examples of finite and infinite dimensional Banach spaces arising from finite dimensional geometry, spaces of sequences, and function spaces.
- Be familiar with the concept of bounded linear operator and with the notion of operator norm. In the exercises, given a linear operator (or a linear functional), the student should be able to determine whether or not the operator is linear and bounded.
- Be able to determine the range and the kernel of a linear operator.
- Be familiar with the concept of closed range operator.
- Be able to prove that all norms are equivalent in a finite dimensional normed space.
- Be familiar with the concepts of uniform convergence and strong convergence for sequences of bounded linear operators.
- Be familiar with the notion of compact operator.
- Be familiar with the notion of dual space, being able to determine dual spaces of the main Banach spaces considered in the course.
- Be familiar with the notion of weak convergence on a Banach space, in particular on L^p and ℓ^p spaces. Especially in L^p, the student should be familiar with the three most important phenomena in which weak convergence occurs and strong convergence does not. Given a sequence in a Banach space, the student should be able to determine whether or not the sequence converges weakly on that space. The student should know the main results on weak compactness on infinite dimensional spaces.

5 Hilbert spaces

Hilbert spaces are Banach spaces with a norm that is derived from an inner product, so they have an extra feature in comparison with arbitrary Banach spaces, which makes them still more special. We can use the inner product to introduce the notion of orthogonality in a Hilbert space, and the geometry of Hilbert spaces is in almost complete agreement with our intuition of linear spaces with an arbitrary (finite or infinite) number of orthogonal coordinate axes. By contrast, the geometry of infinite-dimensional Banach spaces can be surprisingly complicated and quite different from what naive extrapolations of the finite-dimensional situation would suggest.

5.1 *Inner products*

To be specific, we consider complex linear spaces throughout this section. We use a bar to denote the complex conjugate of a complex number. The corresponding results for real linear spaces are obtained by replacing $\mathbb{C}$ by $\mathbb{R}$ and omitting the complex conjugates.

5.1 DEFINITION. An *inner product* on a complex linear space X is a map

$$(\cdot, \cdot) : X \times X \to \mathbb{C}$$

such that, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$:

(a) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (linear in the second argument);

(b) $(y, x) = \overline{(x, y)}$ (Hermitian symmetric);

(c) $(x, x) \geq 0$ (nonnegative);

(d) $(x, x) = 0$ if and only if $x = 0$ (positive definite).

We call a linear space with an inner product an *inner product space* or a *pre-Hilbert space*.

From (a) and (b) it follows that $(\cdot, \cdot)$ is *antilinear*, or *conjugate linear*, in the first argument, meaning that

$$(\lambda x + \mu y, z) = \overline{\lambda}(x, z) + \overline{\mu}(y, z).$$

If X is real, then $(\cdot, \cdot)$ is bilinear, meaning that it is a linear function of each argument. If X is complex, then $(\cdot, \cdot)$ is said to be *sesquilinear*.

There are two conventions for the linearity of the inner product. In most of the mathematically oriented literature $(\cdot, \cdot)$ is linear in the first component. We adopt the convention that the inner product is linear in the second component, which is more common in applied mathematics and physics.

If X is a linear space with an inner product $(\cdot, \cdot)$, then we can define a norm on X by

$$\|x\| = \sqrt{(x, x)}.$$

To see that the above $\|\cdot\|$ is actually a norm, due to the properties (a)-(d) above we only need to prove the triangle inequality. Such a property follows form the following one.

5.2 theorem (Cauchy-Schwarz inequality). *Let X be an inner product space, and let $x, y \in X$. Then*

$$|(x,y)| \leq \|x\|\|y\|. \tag{48}$$

Proof. By the nonnegativity of the inner product we have

$$0 \leq (\lambda x - \mu y, \lambda x - \mu y)$$

for all $x, y \in X$ and $\lambda, \mu \in \mathbb{C}$. Expansion of the inner product implies

$$\overline{\lambda}\mu(x,y) + \lambda\overline{\mu}(y,x) \leq |\lambda|^2\|x\|^2 + |\mu|^2\|y\|^2.$$

If $(x,y) = re^{i\varphi}$, where $r = |(x,y)|$ and $\varphi = \operatorname{Arg}(x,y)$, then we choose

$$\lambda = \|y\|e^{i\varphi}, \qquad \mu = \|x\|.$$

It follows that

$$2\|x\|\|y\||(x,y)| \leq 2\|x\|^2\|y\|^2,$$

which proves the result. $\square$

As a consequence of (48), given $x, y \in X$ we have

$$\begin{aligned} &\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + \|y\|^2 + (x,y) + (y,x) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2, \end{aligned}$$

and this proves the triangle inequality.

5.3 definition. A *Hilbert space* is a complete inner product space.

5.4 example. The standard inner product on $\mathbb{C}^n$ is given by

$$(x,y) = \sum_{j=1}^{n} \overline{x}_j y_j,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, with $x_j, y_j \in \mathbb{C}$. This space is complete, and therefore it is a finite-dimensional Hilbert space.

5.5 example. Let $C([a,b])$ denote the space of all-complex-valued continuous functions defined on the interval $[a,b]$. We define an inner product on $C([a,b])$

by

$$(f,g) = \int_a^b \overline{f(x)} g(x) dx,$$

where $f, g : [a,b] \to \mathbb{C}$ are continuous functions. This space is not complete, so it is not a Hilbert space.

5.6 example. Let $\Omega \subset \mathbb{R}^d$ be an open set. Given $f, g \in L^2(\Omega)$ with complex values, i.e. $f, g : \Omega \to \mathbb{C}$, we define as in the previous example

$$(f,g) = \int_\Omega \overline{f(x)} g(x) dx. \tag{49}$$

Then, it is easily seen that

$$(f,f) = \|f\|^2_{L^2(\Omega)},$$

which proves that the L^2 norm is induced by the inner product (49)[12]. Since L^2 is a complete space, then we have just proven that L^2 is a Hilbert space. L^2 is the only L^p space to be a Hilbert space.

5.7 example. We define the Hilbert space $\ell^2(\mathbb{Z})$ of bi-infinite complex sequences by

$$\ell^2(\mathbb{Z}) = \left\{ (z_n)_{n=-\infty}^{+\infty} \; : \; \sum_{n=-\infty}^{+\infty} |z_n|^2 < +\infty \right\}.$$

The space $\ell^2(\mathbb{Z})$ is a complex linear space, with the obvious operations of addition and multiplication by a scalar. An inner product on it is given by

$$(x,y) = \sum_{n=-\infty}^{+\infty} \overline{x_n} y_n.$$

The space $\ell^2(\mathbb{N})$ of squared-summable sequences $(z_n)_{n=1}^{+\infty}$ is defined in an analogous way. The fact the these spaces are complete follows by the completeness of the $\ell^p(\mathbb{N})$ spaces proven earlier in this course.

5.8 theorem (Parallelogram law). *On an inner product space X we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

for all $x, y \in X$.

[12]The L^p spaces defined in Section 3.4 consist of functions with real values, but the whole L^p theory can be extended for complex valued functions

Proof. We compute

$$
\begin{aligned}
&\|x+y\|^2 + \|x-y\|^2 \\
&= 2\|x\|^2 + 2\|y\|^2 + (x,y) + (y,x) - (x,y) - (y,x) \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}
$$

□

5.9 exercise. Use Cauchy-Schwarz inequality to prove that the inner product is a continuous function on an inner product space with respect to both components.

5.2 *Orthogonality*

Let H be a Hilbert space. We denote its inner product by $\langle \cdot, \cdot \rangle$, which is another common notation for inner products that is often reserved for Hilbert spaces. The inner product structure of a Hilbert space allows us to introduce the concept of orthogonality, which makes it possible to visualize vectors and linear subspaces of a Hilbert space in a geometric way.

5.10 definition. If x, y are vectors in a Hilbert space H, then we say that x and y are *orthogonal*, written $x \perp y$, if $\langle x, y \rangle = 0$. We say that subsets A and B are orthogonal, written $A \perp B$, if $x \perp y$ for every $x \in A$ and $y \in B$. The *orthogonal complement* $A^\perp$ of a subset A is the set of vectors orthogonal to A,

$$
A^\perp = \{x \in H \,:\, x \perp y \text{ for all } y \in A\}.
$$

5.11 theorem. *The orthogonal complement of a subset of a Hilbert space is a closed linear subspace.*

Proof. Let H be a Hilbert space and A a subset of H. If $x, y \in A^\perp$ and $\lambda, \mu \in \mathbb{C}$, then the linearity of the inner product implies that

$$
\langle x, \lambda y + \mu z \rangle = \lambda \langle x + y \rangle + \mu \langle x, z \rangle = 0
$$

for all $x \in A$. Therefore, $\lambda y + \mu z \in A^\perp$, so $A^\perp$ is a linear subspace.

To show that $A^\perp$ is closed, we show that if (y_n) is a convergent sequence in $A^\perp$, then the limit y also belongs to $A^\perp$. Let $x \in A$. From the continuity of the inner product we have

$$
\langle x, y \rangle = \langle x, \lim_{n \to +\infty} y_n \rangle = \lim_{n \to +\infty} \langle x, y_n \rangle = 0,
$$

since $\langle x, y_n \rangle = 0$ for every $x \in A$ and $y_n \in A^\perp$. Hence, $y \in A^\perp$. □

The following theorem expresses one of the fundamental geometrical properties of Hilbert spaces. While the result may appear obvious, the proof is not trivial.

5.12 THEOREM (Orthogonal Projection). *Let M be a closed linear subspace of a Hilbert space H.*

(a) *For each $x \in H$ there is a unique closest point $y \in M$ such that*

$$\|x - y\| = \min_{z \in M} \|x - z\|.$$

(b) *The point $y \in M$ closest to $x \in H$ is the unique element of M with the property that $(x - y) \perp M$.*

Proof. Let d be the distance of x from M,

$$d = \inf\{\|x - y\| \ : \ y \in M\}.$$

First, we prove that there is a closest point $y \in M$ at which this infimum is attained, meaning that $\|x - y\| = d$. From the definition of d, there is a sequence of elements $y_n \in M$ such that

$$\lim_{n \to +\infty} \|x - y_n\| = d.$$

Thus, for all $\epsilon > 0$, there is an N such that

$$\|x - y_n\| \le d + \epsilon \qquad \text{when } n \ge N.$$

We show that the sequence (y_n) is Cauchy. From the parallelogram law, we have

$$\|y_m - y_n\|^2 + \|2 - y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2.$$

Since $(y_n + y_m)/2 \in M$, the definition of d implies

$$\|x - (y_m + y_n)/2\| \ge d.$$

Hence, for all $m, n \ge N$, we get

$$\begin{aligned} &\|y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2 - y_m - y_n\|^2 \\ &\quad \le 4(d + \epsilon)^2 - 4d^2 = 4\epsilon(2d + \epsilon). \end{aligned}$$

Therefore, (y_n) is Cauchy. Since a Hilbert space is complete, there is a y such that $y_n \to y$, and since M is closed, we have $y \in M$. The norm is continuous, so $\|x - y\| = \lim_{n \to +\infty} \|x - y_n\| = d$.

Second, we prove the uniqueness of a vector $y \in M$ that minimizes $\|x - y\|$. Suppose y and y' both minimize the distance to x, meaning that

$$\|x - y\| = \|x - y'\| = d.$$

Then the parallelogram law implies that

$$2\|x - y\|^2 + 2\|x - y'\|^2 = \|2x - y - y'\|^2 + \|y - y'\|^2.$$

Hence, since $(y + y')/2 \in M$,

$$\|y - y'\|^2 = 4d^2 - 4\|x - (y + y')/2\|^2 \leq 0,$$

therefore $\|y - y'\| = 0$ so that $y = y'$.

Third, we show that the unique $y \in M$ found above satisfies the condition that the 'error' vector $x - y$ is orthogonal to M. Since y minimizes the distance to x, we have for every $\lambda \in \mathbb{C}$ and $z \in M$ that

$$\|x - y\|^2 \leq \|x - y - \lambda z\|^2.$$

Expanding the right-hand side of this equation, we obtain that

$$2\mathrm{Re}\lambda \langle x - y, z\rangle \leq |\lambda|^2 \|z\|^2.$$

Suppose that $\langle x - y, z\rangle = |\langle x - y, z\rangle| e^{i\varphi}$. Choosing $\lambda = \epsilon e^{-i\varphi}$, where $\epsilon > 0$, and dividing by ϵ, we get

$$2|\langle x - y, z\rangle| \leq \epsilon \|z\|^2.$$

Taking the limit as $\epsilon \to 0^+$, we find that $\langle x - y, z\rangle = 0$, so $(x - y) \perp M$.

Finally, we show that y is the only element in M such that $x - y \perp M$. Suppose that y' is another such element in M. Then $y - y' \in M$, and, for any $z \in M$ we have

$$\langle z, y - y'\rangle = \langle z, x - y'\rangle - \langle z, x - y\rangle = 0.$$

In particular, we may take $z = y - y'$, and therefore we must have $y = y'$. $\square$

The point $y \in M$ above is called the *Orthogonal Projection* of x onto M.

The proof of part (a) applies if M is any closed convex subset of H (exercise). Theorem 5.12 can also be stated in terms of decomposition of H into an *orthogonal direct sum* of closed subspaces.

5.13 definition. If M and N are orthogonal closed linear subspaces of a Hilbert space H, then we define the *orthogonal direct sum*, or simply *direct sum*, $M \oplus N$ of M and N by

$$M \oplus N = \{y + z \,:\, y \in M \text{ and } z \in N\}.$$

Theorem 5.12 states that if M is a closed subspace, then any $x \in H$ may be uniquely represented as $x = y + z$, where $y \in M$ is the best approximation to x and $z \perp M$. We therefore have the following corollary.

5.14 corollary. *If M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^\perp$.*

Thus, every closed linear subspace M of a Hilbert space has a closed complementary subspace $M^\perp$. If M is not closed, then we may still decompose H as $H = \overline{M} \oplus M^\perp$. In a general Banach space, there may be no element of a closed subspace that is closest to a given element of the Banach space.

5.3 *Orthonormal bases*

A subset U of nonzero vectors in a Hilbert space H is *orthogonal* if any two distinct elements in U are orthogonal. A set of vectors U is *orthonormal* if it is orthogonal and $\|u\| = 1$ for all $u \in U$, in which case the vectors u are said to be *normalized*. An *orthonormal basis* of a Hilbert space is an orthonormal set such that every vector in the space can be expanded in terms of the basis, in a way that we make precise below.

In this section we show that every Hilbert space has an orthonormal basis, which may be finite, countably finite, of uncountable. Two Hilbert spaces whose orthonormal bases have the same cardinality are isomorphic, but many different concrete realizations of a given abstract Hilbert space arise in applications. The most important case in practice is that of a *separable Hilbert space*, which has a finite of countably infinite orthonormal basis. As shown below, this condition is equivalent to the separability of the Hilbert space as a metric space, meaning that it contains a countable dense subset.

5.15 example. A set of vectors $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the finite-dimensional Hilbert space $\mathbb{C}^n$ if:

(a) $\langle e_j, e_k \rangle = \delta_{jk}$ for $1 \leq j, k \leq n$;

(b) for all $x \in \mathbb{C}^n$ there are unique coordinates $x_k \in \mathbb{C}$ such that

$$x = \sum_{k=1}^{n} x_k e_k,$$

where δ_{jk} is the *Kronecker delta* symbol

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

The orthonormality of the basis implies that $x_k = \langle e_k, x \rangle$. For example, the standard orthonormal basis of $\mathbb{C}^n$ consists of the vectors

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad e_n = (0, 0, \ldots, 1).$$

5.16 example. Consider the Hilbert space $\ell^2(\mathbb{Z})$ defined in example 5.7. An orthonormal basis of $\ell^2(\mathbb{Z})$ is the set of coordinate basis vectors $\{e_n : n \in \mathbb{Z}\}$ given by

$$e_n = (\delta_{kn})_{k=-\infty}^{+\infty}.$$

5.17 example. The set of functions $\{e_n : n \in \mathbb{Z}\}$, given by

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx},$$

is an orthonormal basis of the space $L^2(\mathbb{T})$ of 2π-periodic functions, called the *Fourier basis*. We will study it in detail below. As we will see, the map $\mathcal{F}^{-1} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ defined by

$$\mathcal{F}^{-1}((c_k)_k) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k e^{ikx}$$

is a Hilbert space isomorphism between $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$. Such a map is called *inverse Fourier transform*. Both Hilbert spaces are separable with a countably infinite basis.

5.18 THEOREM (Bessel's inequality). *Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal sequence in a Hilbert space H and $x \in H$. Then,*

(a) $\sum_{n\in\mathbb{N}} |\langle u_n, x\rangle|^2 \le \|x\|^2$;

(b) $x_U \doteq \sum_{n=1}^{+\infty} \langle u_n, x\rangle u_n$ *is a convergent sum;*

(c) $x - x_U \in U^\perp$.

Proof. We begin by computing $\|x - \sum_{n=1}^N \langle u_n, x\rangle u_n\|$ for any $N \in \mathbb{N}$:

$$\begin{aligned}\left\| x - \sum_{n=1}^N \langle u_n, x\rangle u_n \right\|^2 &= \left\langle \left(x - \sum_{n=1}^N \langle u_n, x\rangle u_n \right), \left(x - \sum_{m=1}^N \langle u_m, x\rangle u_m \right) \right\rangle \\ &= \langle x, x\rangle - \sum_{m=1}^N \langle u_m, x\rangle \langle x, u_m\rangle - \sum_{n=1}^N \overline{\langle u_n, x\rangle} \langle u_n, x\rangle + \sum_{n,m=1}^N \overline{\langle u_n, x\rangle} \langle u_m, x\rangle \langle u_n, u_m\rangle \\ &= \|x\|^2 - \sum_{n=1}^N |\langle u_n, x\rangle|^2.\end{aligned}$$

Hence,

$$\sum_{n=1}^N |\langle u_n, x\rangle|^2 = \|x\|^2 - \left\| x - \sum_{n=1}^N \langle u_n, x\rangle u_n \right\|^2 \le \|x\|^2.$$

Since $\sum_{n=1}^N |\langle u_n, x\rangle|^2$ is a sum of nonnegative numbers that is bounded from above by $\|x\|^2$, it is the partial sum of a convergent series. Therefore the sum converges and satisfies (a). The convergence claimed in (b) follows by the fact that, for given $N, M \in \mathbb{N}$, one has

$$\left\| \sum_{n=M+1}^N u_n \right\|^2 = \sum_{n,m=M+1}^N \langle u_n, u_m\rangle = \sum_{n=M+1}^N \|u_n\|^2.$$

Now, since the right hand side goes to zero as $N, M \to +\infty$ because of (a), then the left hand side is infinitesimal for large N, M. In order to prove (c), we consider any $u_{k_0} \in U$. Using the orthonormality of U and the continuity of the inner product, we find that

$$\left\langle x - \sum_{n=1}^{+\infty} \langle u_n, x\rangle u_n, u_{k_0} \right\rangle$$
$$= \langle x, u_{k_0}\rangle - \sum_{n=1}^{+\infty} \langle u_n, x\rangle \langle u_n, u_{k_0}\rangle = \langle x, u_{k_0}\rangle - \langle x, u_{k_0}\rangle = 0.$$

Hence, $x - \sum_{n=1}^{+\infty} \langle u_n, x\rangle u_n \in U^{\perp}$. □

Given a subset $U \subset H$, we define the closed linear span $[U]$ of U by

$$[U] = \left\{ \sum_{u \in U} c_u u \ : \ c_u \in \mathbb{C} \text{ and } \textstyle\sum_{u \in U} c_u u \text{ converges} \right\}.$$

Equivalently, $[U]$ is the smallest closed linear subspace that contains U. We leave the proof of the following lemma to the student.

5.19 lemma. *If $U = \{u_n \ : \ n \in \mathbb{N}\}$ is an orthonormal set in a Hilbert space H, then*

$$[U] = \left\{ \sum_{n=1}^{+\infty} c_n u_n \ : \ c_n \in \mathbb{C} \text{ and } \textstyle\sum_{n=1}^{+\infty} |c_n|^2 < +\infty \right\}.$$

By combining Theorem 5.12 and Theorem 5.18 we see that x_U, defined in part (c) of Theorem 5.18, is the unique element of $[U]$ satisfying

$$\|x - x_U\| = \min_{u \in [U]} \|x - u\|.$$

In particular, if $[U] = H$, then $x_U = x$, and every $x \in H$ may be expanded in terms of elements of U. The following theorem gives equivalent conditions for this property of U, called *completeness*.

5.20 theorem. *If $U = \{u_n \ : \ n \in \mathbb{N}\}$ is an orthonormal sequence of a Hilbert space H, then the following conditions are equivalent:*

(a) $\langle u_n, x\rangle = 0$ *for all* $n \in \mathbb{N}$ *implies* $x = 0$*;*

(b) $x = \sum_{n \in \mathbb{N}} \langle u_n, x\rangle u_n$ *for all* $x \in H$*;*

(c) $\|x\|^2 = \sum_{n \in \mathbb{N}} |\langle u_n, x\rangle|^2$ *for all* $x \in H$*;*

(d) $[U] = H$*;*

(e) U *is a maximal orthonormal sequence.*

Proof. The condition (a) states that $U^{\perp} = \{0\}$. Part (c) of Theorem 5.18 the implies (b). The fact that (b) implies (c) follows from the same computation used to prove (b) in Theorem 5.18. To prove that (c) implies (d), we observe that (c) implies that $U^{\perp} = \{0\}$, which implies that $[U]^{\perp} = \{0\}$, so $[U] = H$. Condition (e) means that if V is a subset of H that contains U and is strictly larger than U, than V is not orthonormal. To prove that (d) implies (e), we note from (d) that any $v \in H$ is of the form $v = \sum_{n\in\mathbb{N}} c_n u_n$, where $c_n = \langle u_n, x\rangle$. Therefore, if $v \perp U$ then $c_n = 0$ for all $n \in \mathbb{N}$, and hence $v = 0$, so $U \cup \{v\}$ is not orthonormal. Finally, (e) implies (a), since (a) is just a reformulation of (e). This proves the theorem (all the statements are equivalent). $\square$

In view of this theorem, we may now introduce the following definition.

5.21 definition. An orthonormal set $U = \{x_n : n \in \mathbb{N}\}$ of a Hilbert space H is *complete* if it satisfies any of the equivalent conditions (a)-(e) in Theorem 5.20. A complete orthonormal subset of H is also called an *orthonormal basis* of H.

Condition (a) is often the easiest to verify. Condition (b) is the property that is used most often. Condition (c) is a special case of the so-called *Parseval's identity* (see below). Condition (d) simply expresses completeness of the basis, and condition (e) expresses the factg that an orthonormal bases cannot be extended by adding one more vector in a way to still get an independent set of vectors.

The following identity shows that a Hilbert space H with orthonormal basis $\{u_n : n \in \mathbb{N}\}$ is isomorphic to the sequence space $\ell^2(\mathbb{Z})$. The proof is left to the student.

5.22 theorem (Parseval's identity). *Suppose that $U = \{u_n : n \in \mathbb{N}\}$ is an orthonormal basis of a Hilbert space H. If $x = \sum_{n\in\mathbb{N}} a_n u_n$ and $y = \sum_{n\in\mathbb{N}} b_n u_n$, where $a_n = \langle u_n, x\rangle$ and $b = \langle u_n, y\rangle$, then*

$$\langle x, y\rangle = \sum_{n\in\mathbb{N}} \overline{a_n} b_n.$$

Orthonormal basis play an essential role in Hilbert spaces. It can be proven (but we shall omit the proof) that an arbitrary Hilbert space can be equipped with an orthonormal basis. In general, this basis may be countable or more than countable. One can prove that a separable Hilbert space always has a countable orthonormal basis. The procedure used to prove that is the so-called *Gram-Schmidt orthonormalization* procedure, i.e. an algorithm for the construction of an orthonormal basis from any countable linearly independent set whose linear span is dense in H. We omit the details and state this important assertion as a Theorem. In fact, the existence of an orthonormal basis is equivalent to the separability of the Hilbert space.

5.23 theorem. *A Hilbert space is separable if and only if it has a countable orthonormal basis.*

What makes Hilbert spaces so powerful in many applications is the possibility of expressing a problem in terms of a suitable orthonormal basis. Here we consider *Fourier series*, which corresponds to the expansion of periodic functions with respect to an orthonormal basis of trigonometric functions.

A function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if

$$f(x + 2\pi) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

The choice of the 2π for the period is simply for convenience; different periods may be reduced to this case by rescaling the independent variable. A 2π-periodic function on $\mathbb{R}$ may be identified with a function on the circle, or one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, which we identify by identifying points in $\mathbb{R}$ that differ by $2\pi n$ for some $n \in \mathbb{Z}$. We could instead represent a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ by a function on a closed interval $f : [a, a + 2\pi] \to \mathbb{C}$ such that $f(a) = f(a + 2\pi)$, but the choice of a here is arbitrary, and it is clearer to think of the function as defined on the circle, rather than an interval.

The space $C(\mathbb{T})$ is the space of continuous functions from $\mathbb{T}$ to $\mathbb{C}$, and $L^2(\mathbb{T})$ is the closure of $C(\mathbb{T})$ with respect to the L^2 norm

$$\|f\|_{L^2} = \left(\int_{\mathbb{T}} |f(x)|^2 dx \right)^{1/2}.$$

Here, the integral over $\mathbb{T}$ is an integral with respect to x taken over any interval of length 2π. We recall that $L^2(\mathbb{T})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} \overline{f(x)} g(x) dx.$$

The Fourier basis elements are the functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

We recall by Euler's formula that

$$e^{inx} = \cos(nx) + i \sin(nx).$$

Our first objective is to prove that $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{T})$. The orthonormality of the functions e_n is a simple computation:

$$\begin{aligned}
\langle e_m, e_n \rangle &= \int_{\mathbb{T}} \overline{\frac{1}{\sqrt{2\pi}} e^{imx}} \frac{1}{\sqrt{2\pi}} e^{inx} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)x} dx \\
&= \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
\end{aligned}$$

Thus, the main result that we have to prove is the completeness of $\{e_n : n \in \mathbb{N}\}$. We denote the set of all finite linear combination of the e_n by $\mathcal{P}$. Functions in $\mathcal{P}$ are called *trigonometric polynomials*. We will prove that any continuous function on $\mathbb{T}$ can be approximated uniformly by trigonometric polynomials. Since uniform convergence on $\mathbb{T}$ implies L^2-convergence, and continuous functions are dense in $L^2(\mathbb{T})$, it follows that the trigonometric polynomials are dense in $L^2(\mathbb{T})$, so $\{e_n\}$ is a basis.

5.24 theorem. *The trigonometric polynomials are dense in $C(\mathbb{T})$ with respect to the uniform norm.*

Proof. For each $n \in \mathbb{N}$, we define the function $\varphi_n \geq 0$ by

$$\varphi_n(x) = c_n(1+\cos x)^n.$$

We choose the constants c_n so that

$$\int_{\mathbb{T}} \varphi_n(x)dx = 1.$$

Since $1+\cos x$ has a strict maximum at $x = 0$, the graph of φ_n is sharply peaked at $x = 0$ for large n, and the area under the graph concentrates near $x = 0$. In particular, φ_n satisfies

$$\lim_{n\to+\infty} \int_{\delta\leq|x|\leq\pi} \varphi_n(x) = 0 \qquad \text{for every } \delta > 0. \tag{50}$$

In order to prove (50), we observe that for all $x \in [-\pi, \pi]$ with $\delta < |x| < \pi$ we have

For simplicity we shall just work on the interval $[0, 2\pi]$, and the result follows by periodicity. We shall use an idea suggested by *Cesàro* in the nineteenth century. Let $u \in C([0, 2\pi])$. We consider the *truncated Fourier series*

$$U_n(x) = \frac{1}{2\pi} \sum_{|k|\leq n} \left(\int_0^{2\pi} u(t)e^{-ikt}dt \right) e^{ikx}.$$

This is just a finite sum, so we can treat x as a parameter and use the linearity of the integral to write this as

$$U_n(x) = \int_0^{2\pi} D_n(x-t)u(t)dt, \qquad D_n(s) = \frac{1}{2\pi} \sum_{|k|\leq n} e^{iks}.$$

Now this sum can be written as an explicit quotient, since, by telescoping

$$2\pi D_n(s)(e^{is/2} - e^{-is/2}) = e^{i(n+1/2)s} - e^{-i(n+1/2)s}.$$

So in fact, at least when $s \neq 0$,

$$D_n(s) = \frac{e^{i(n+1/2)s} - e^{-i(n+1/2)s}}{2\pi(e^{is/2} - e^{-is/2})}.$$

A simple computation shows that in fact $D_n(s)$ can be extended continuously to $s = 0$ by setting

$$D_n(0) = \frac{n + \frac{1}{2}}{\pi}.$$

Cesàro's idea is to speed up the convergence of U_n to u by replacing the U_n's by their averages. We set

$$V_n(x) = \frac{1}{n+1} \sum_{l=1}^{n} U_l.$$

Using the previous computations, we get

$$V_n(x) = \int_0^{2\pi} S_n(x-t)u(t)dt, \quad S_n(s) = \frac{1}{n+1} \sum_{l=1}^{n} D_l(s).$$

Once again, we want to compute a more useful form of V_n. We compute

$$2(n+1)\pi S_n(s)(e^{is/2} - e^{-is/2}) = \sum_{l=0}^{n} e^{i(l+1/2)s} - \sum_{l=0}^{n} e^{-i(l+1/2)s}.$$

Using the same trick again,

$$(e^{is/2} - e^{-is/2}) \sum_{l=0}^{n} e^{i(l+1/2)s} = e^{i(n+1)s} - 1,$$

and

$$(e^{is/2} - e^{-is/2}) \sum_{l=0}^{n} e^{-i(l+1/2)s} = 1 - e^{-i(n+1)s},$$

we obtain

$$2(n+1)\pi S_n(s)(e^{is/2} - e^{-is/2})^2 = e^{i(n+1)s} + e^{-i(n+1)s} - 2,$$

which implies

$$S_n(s) = \frac{e^{i(n+1)s} + e^{-i(n+1)s} - 2}{2\pi(n+1)(e^{is/2} - e^{-is/2})^2} = \frac{1}{n+1} \frac{\sin^2\left(\frac{n+1}{2}s\right)}{2\pi \sin^2\left(\frac{s}{2}\right)}.$$

The function S_n is called *Fejér kernel*. One thing which is immediately clear from the above discussion is that if we plug $u = 1$ we get $U_n = 1$ for all $n \geq 0$, and hence $V_n = 1$ for all $n \geq 0$. Consequently,

$$\int_0^{2\pi} S_n(x-t)dt = 1 \qquad \text{for all } x \in \mathbb{R}. \tag{51}$$

Moreover, $S_n \geq 0$ everywhere. The denominator of S_n only vanishes on the multiples on $x = 0, 2\pi$. The limit of S_n on those points can be computed to be $(n+1)/2\pi$. Moreover, if we stay away from $x = 0, 2\pi$, then S_n converges to zero uniformly. More precisely, let $\delta > 0$ be fixed, and consider $x \in [\delta, 2-\delta]$. We get the estimate

$$\max_{x \in [\delta, 2-\delta]} S_n(x) \leq \frac{1}{2\pi(n+1)\sin^2(\delta/2)} \to 0 \quad \text{as } n \to +\infty. \tag{52}$$

Now, we are interested in how close V_n gets to u as $n \to +\infty$. Due to (51), we easily get

$$V_n(x) - u(x) = \int_0^{2\pi} S_n(x-t)(u(t) - u(x))dt.$$

We split the above integral into two parts and use the triangle inequality to get

$$|V_n(x) - u(x)| \leq \int_{\{|x-t| \in [0,\delta) \cup (2\pi-\delta, 2\pi]\}} S_n(x-t)|u(t) - u(x)|dt$$
$$+ \int_{\{|x-t| \in [\delta, 2\pi-\delta]\}} S_n(x-t)|u(t) - u(x)|dt.$$

Since u is uniformly continuous, for a given $\epsilon > 0$ there is a $\delta > 0$ such that $|u(t) - u(x)| \leq \epsilon/2$ anytime $|x-t| < \delta$. Consider that the the interval $[2\pi - \delta, 2\pi]$ has to be computed carefully using the periodicity assumptions, the details are left to the student. Hence, the first integral term above is controlled by $\epsilon/2 \int_0^{2\pi} S_n(x,t)dt = \epsilon/2$. On the other integral term, we can use the estimate (52) and the boundedness of u (u is continuous on a compact interval). We then obtain

$$|V_n(x) - u(x)| \leq \epsilon/2 + 4\pi\|u\|_\infty \frac{1}{2\pi(n+1)\sin^2(\delta/2)}.$$

Hence, it is easy to find an $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$ one gets

$$\sup_{x \in [0,2\pi]} |V_n(x) - u(x)| \leq \epsilon,$$

i.e. V_n converges uniformly to u. Since D_n is a linear combination of trigonometric functions and S_n is a linear combination of terms of the form D_n, then V_n is a trigonometric polynomial by definition. The proof is complete. $\square$

The result in Theorem 5.24 and the density of continuous functions in L^2 imply that trigonometric polynomials are dense in L^2. Hence, the set $\mathcal{U} = \{e_n : n \in \mathbb{Z}\}$ spans the whole Hilbert space $L^2(\mathbb{T})$. Due to Theorem 5.20, this implies that $\mathcal{U}$ is an orthonormal basis for $L^2(\mathbb{T})$. This means that any function $f \in L^2(\mathbb{T})$ can be expanded in a Fourier series as

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{f}_n e^{inx}, \tag{53}$$

where

$$\hat{f}_n = \langle e_n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx.$$

The identity (53) means convergence of the partial sums of f in the L^2-norm, i.e.

$$\lim_{N \to +\infty} \int_{\mathbb{T}} \left| \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx} - f(x) \right|^2 dx = 0.$$

Moreover, Parseval's identity implies that

$$\int_{\mathbb{T}} \overline{f(x)} g(x) dx = \sum_{n=-\infty}^{+\infty} \overline{\hat{f}_n} \hat{g}_n.$$

In particular, the L^2-norm of an $L^2(\mathbb{T})$-function can be computed either in terms of the function or its Fourier coefficients, since

$$\int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |\hat{f}_n|^2.$$

Thus, the periodic Fourier transform $\mathcal{F}(\mathbb{T}) \to \ell^2(\mathbb{Z})$ that maps a function to its sequence of Fourier coefficients, by

$$\mathcal{F} f = \left(\hat{f}_n \right)_{n=-\infty}^{+\infty},$$

is a Hilbert space isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. The projection theorem, Theorem 5.12, implies that the partial sum

$$F_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}_n e^{inx}$$

is the best approximation of f by a trigonometric polynomial of degree N in the sense of the L^2-norm.

The L^2 convergence of the Fourier series is particularly simple. It is nevertheless interesting to ask about other types of convergence. For example, the Fourier series of a function $f \in L^2(\mathbb{T})$ also converges pointwise almost everywhere to f. This result was proven by Carleson, only as recently as 1966. An analysis of the pointwise convergence of Fourier series is very subtle, and the proof is beyond the scope of these notes. For smooth functions, such as continuously differentiable functions, the convergence of the partial sums is uniform. We omit the details.

The behaviour of the partial sums near a point of discontinuity of a piecewise smooth function is interesting. The sums to not converge uniformly; instead the partial sums oscillate in an interval that contains the point of discontinuity. The width of the interval where the oscillation occur shrinks to zero as $N \to +\infty$, but the size of the oscillations does not. This behaviour is called the *Gibbs phenomenon*.

We conclude this section with some examples of orthogonal projections on closed subspaces M of a Hilbert space, that is maps $P : H \to M$ provided by the projection theorem.

5.25 EXAMPLE. The space $L^2(\mathbb{R})$ is the orthogonal direct sum of the space M of even functions and the space N of odd functions. The orthogonal projections P and Q of H onto M and N, respectively, are given by

$$Pf(x) = \frac{f(x) - f(-x)}{2}, \qquad Q(x) = \frac{f(x) - f(-x)}{2}.$$

Note that $\mathbb{I} - P = Q$.

5.26 EXAMPLE. Suppose that A is a measurable subset of $\mathbb{R}$ - for example, an interval - with characteristic function

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then

$$P_A f(x) = 1_A(x) f(x)$$

is an orthogonal projection of $L^2(\mathbb{R})$ onto the subspace of functions with support contained in $\overline{A}$.

A frequently encountered case is that of projections onto a one-dimensional subspace of a Hilbert space H. For any vector $u \in H$ with $\|u\| = 1$, the map P_u defined by

$$P_u x = \langle u, x \rangle u$$

projects a vector orthogonally onto its component in the direction u.

5.27 EXAMPLE. If $H = \mathbb{R}^n$, the orthogonal projection P_u in the direction of a unit vector u has the rank one matrix uu^T. The component of a vector x in the direction u is $P_u x = (u^T x)u$.

5.28 EXAMPLE. If $H = \ell^2(\mathbb{Z})$, and $u = e_n$, where $e_n = (\delta_{k,n})_{k=-\infty}^{+\infty}$, and $x = (x_k)$, then $P_{e_n} x = x_n e_n$.

5.29 EXAMPLE. If $H = L^2(\mathbb{T})$ is the space of 2π-periodic functions and $u =$

$1/\sqrt{2\pi}$ is the constant function with norm one, then the orthogonal projection P_u maps a function to its mean: $P_u f = \langle f \rangle$, where

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx.$$

The corresponding orthogonal decomposition,

$$f(x) = \langle f \rangle + \tilde{f}(x),$$

decomposes a function into a constant mean part $\langle f \rangle$ and a fluctuating part $\tilde{f}$ with zero mean.

5.4 *The dual of a Hilbert space*

A *linear functional* on a complex Hilbert space H is a linear map from H to $\mathbb{C}$. A linear functional φ is bounded, or continuous, if there exists a constant M such that

$$|\varphi(x)| \leq M\|x\| \qquad \text{for all } x \in H. \tag{54}$$

The norm of a bounded linear functional φ is

$$\|\varphi\| = \sup_{\|x\|\leq 1} |\varphi(x)|. \tag{55}$$

If $y \in H$, then

$$\varphi_y(x) = \langle y, x \rangle \tag{56}$$

is a bounded linear functional on H, with $\|\varphi_y\| = \|y\|$.

5.30 EXAMPLE. Suppose that $H = L^2(\mathbb{T})$. Then, for each $n \in \mathbb{Z}$, the functional $\varphi_n : L^2(\mathbb{T}) \to \mathbb{C}$,

$$\varphi_n(f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} dx,$$

that maps a function to its nth Fourier coefficient is a bounded linear functional. We have $\|\varphi_n\| = 1$ for every $n \in \mathbb{Z}$.

One of the fundamental facts about Hilbert spaces is that all bounded linear functionals are of the form (56).

5.31 THEOREM (Riesz' representation theorem). *Let H be a Hilbert space, and let $f \in H'$ a linear and continuous functional on H. Then, there exists a unique $z \in H$ such that*

$$\langle f, x \rangle = (z, x), \qquad \textit{for all } x \in H. \tag{57}$$

The map $\sigma : H' \ni f \mapsto z \in H$ is a bijection of H' onto H, it is an isometry, i. e. $\|\sigma(f)\|_H = \|f\|_{H'}$, and it is anti-linear, i. e. $\sigma(f + \lambda g) = \sigma(f) + \overline{\lambda}\sigma(g)$ for all $f, g \in H'$ and all $\lambda \in \mathbb{C}$.

Proof. Let $N = \operatorname{Ker}(f)$. If $N = H$, then $f \equiv 0$, and we set $z = 0$. If $N \neq H$ then there exists $z_0 \in N^\perp$ with $z_0 \neq 0$: indeed, for a given $x_0 \in H \setminus N$, take $z_0 = Q(x_0)$ with Q the orthogonal projection onto $N^\perp$. We observe that $\langle f, z_0\rangle \neq 0$ as $z_0 \notin \operatorname{Ker}(f)$. Moreover, for all $x \in H$ one has

$$x - \frac{z_0}{\langle f, z_0\rangle}\langle f, x\rangle \in N.$$

Indeed,

$$\langle f, x - \frac{z_0}{\langle f, z_0\rangle}\langle f, x\rangle\rangle = \langle f, x\rangle - \frac{\langle f, z_0\rangle}{\langle f, z_0\rangle}\langle f, x\rangle = 0.$$

Therefore,

$$0 = (z_0, x - \frac{z_0}{\langle f, z_0\rangle}\langle f, x\rangle) = (z_0, x) - \frac{\langle f, x\rangle}{\langle f, z_0\rangle}(z_0, z_0) = (z_0, x) - \|z_0\|^2\frac{\langle f, x\rangle}{\langle f, z_0\rangle},$$

which implies

$$\langle f, x\rangle = \frac{\langle f, z_0\rangle}{\|z_0\|^2}(z_0, x) = (\overline{\frac{\langle f, z_0\rangle}{\|z_0\|^2}}z_0, x).$$

Hence, $\frac{\langle f, z_0\rangle}{\|z_0\|^2}z_0 = \sigma(f)$ is the desired element in H for which (57) holds.

Now, we claim that there is just one vector $z \in H$ with the property (57). Assume $\langle f, x\rangle = (z', x)$ for all $x \in H$. Then, $(z - z', x) = 0$ for all $x \in H$, which implies $z = z'$. Moreover, σ is a bijection, because if $\sigma(f_1) = \sigma(f_2)$ then

$$\langle f_1, x\rangle = (\sigma(f_1), x) = (\sigma(f_2), x) = \langle f_2, x\rangle,$$

for all $x \in H$, i. e. f_1 and f_2 coincide. The anti-linearity is an easy exercise. Let us prove that σ is an isometry. It suffices to prove that $\|\sigma(f)\|_H = \|f\|_{H^*}$. Let $x \in H$, we have (with $z = \sigma(f)$)

$$|\langle f, x\rangle| = |(z, x)| \leq \|x\|\|z\|,$$

which implies $\|f\| \leq \|\sigma(f)\|$. Choosing $x = z$ we have $|\langle f, z\rangle| = \|z\|^2$, which proves the assertion. □

5.32 corollary. *Every Hilbert space is reflexive.*

5.5 *Exercises*

1. Let A be a subset of a Hilbert space H. Prove that $A^\perp = \overline{A}^\perp$.

2. Suppose that H_1 and H_2 are Hilbert spaces. Define

$$H_1 \oplus H_2 = \{(x_1, x_2) \,:\, x_1 \in H_1\,,\ x_2 \in H_2\}$$

with the inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}.$$

Prove that $H_1 \oplus H_2$ is a Hilbert space. Find the orthogonal complement of the subspace $\{(x_1, 0) \,:\, x_1 \in H_1\}$.

3. Let $f, g \in H$, H a Hilbert space. Assume that equality holds in Cauchy-Schwartz inequality for f and g, i. e

$$(f, g) = \|f\|\|g\|.$$

Prove that $f = cg$ for some scalar $c \in \mathbb{C}$. Hint: assume first that $\|f\| = \|g\| = 1$ and use Pythagoras' theorem.

4. Let $\eta : [a, b] \to \mathbb{R}$ be a continuous function such that $\eta(t) > 0$ for all $t \in [a, b]$. For two given functions $f, g : [a, b] \to \mathbb{C}$ define the product

$$(f, g)_\eta := \int_a^b f(x)\overline{g(x)}\eta(x)dx.$$

Prove that $(\cdot, \cdot)_\eta$ is a scalar product, and prove that the resulting normed space is a Hilbert space.

5. Prove the the vectors in an orthogonal set are linearly independent.

6. Let $H = L^2(\mathbb{R})$, and set

$$M = \{f \in H \,:\, f(t) = f(-t) \text{ almost everywhere in } \mathbb{R}\}.$$

- Show that M is a closed subspace of H.
- Find an explicit expression for the orthogonal complement $M^\perp$.
- Find an explicit expression for the orthogonal projection of H onto M.

7. Consider the Hilbert space $H = L^2(\mathbb{R}^d\,; \mathbb{R}^d)$ of all vector fields $v : \mathbb{R}^d \to \mathbb{R}^d$ equipped with the scalar product

$$(u, v) = \int_{\mathbb{R}^d} u(x) \cdot v(x)dx.$$

Consider $u(x) = \nabla V(x) \in H$ for some $V \in C^1(\mathbb{R}^d)$ and $v \in H$ such that $\mathrm{div} v = 0$. Prove that u and v are orthogonal.

8. Let $\{x_n \,:\, n \in \mathbb{N}\}$ be a countable orthonormal set in a Hilbert space. Show that the sum $\sum_{n=1}^{+\infty} \frac{x_n}{n}$ converges unconditionally but not absolutely.

9. Prove the following lemma: If $U = \{u_\alpha : \alpha \in I\}$ is an orthonormal set in a Hilbert space H, then

$$[U] = \left\{ \sum_{\alpha \in I} c_\alpha u_\alpha \ : \ c_\alpha \in \mathbb{C} \text{ and } \textstyle\sum_{\alpha \in I} |c_\alpha|^2 < +\infty \right\}.$$

10. Prove that the sets $\{e_n : n \geq 1\}$ defined by

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),$$

and $\{f_n : n \geq 0\}$ defined by

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \qquad f_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx), \qquad \text{for } n \geq 1$$

are both orthonormal bases of $L^2([0, \pi])$.

11. For each of the following functions $f \in L^2([0, \pi])$, find the Fourier coefficients of f with respect to both the bases $\{e_n : n \geq 1\}$ and $\{f_n : n \geq 0\}$ of the previous exercise:

(a) $f(x) = x^2$,

(b) $f(x) = |x|$,

(c) $f(x) = \begin{cases} 1 & x \in [0, \pi/2] \\ 2 & x \in (\pi/2, \pi] \end{cases}$,

(d) $f(x) = 3\sin(4x) - 7\cos(2x)$,

12. * Define the *Legendre polynomials* P_n by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

- Compute the first few Legendre polynomials.
- Show that the Legendre polynomials are orthogonal in $L^2([-1, 1])$, and that they are obtained by Gram-Schmidt orthogonalisation of the monomials.
- Show that

$$\int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}.$$

- Prove that the set

$$\{\sqrt{\frac{2n+1}{2}} P_n, \ n \in \mathbb{N}\}$$

is an orthonormal basis for $L^2([-1, 1])$

13. * Let $H = L^2([0,1])$. We say that $f \in H$ has a *weak derivative* in L^2 if there exists a function $g \in H$ such that

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 f(x)\phi'(x)dx, \qquad \text{for all } \phi \in C_c^1([0,1]).$$

The function g is called the weak derivative of f, and is denoted by Df. We call $H_0^1 \subset H$ the set of all $f \in L^2$ with a weak derivative in L^2.

- Prove that, if $f \in H$ is a continuously differentiable function, then the weak derivative Df coincides almost everywhere with the classical derivative f'.
- Prove that H_0^1 is a dense linear subspace of L^2. Hint: use the fact that C^1 functions are dense in H.
- Equip H_0^1 with the product

$$(f,g)_{H_0^1} := \int_0^1 f(x)g(x)fx + \int_0^1 Df(x)Dg(x)dx.$$

Prove that H_0^1 is a Hilbert space with the above product.

14. Suppose (P_n) is a sequence of orthogonal projections on a Hilbert space H such that

$$\text{Ran}P_{n+1} \supset \text{Ran}P_n, \qquad \bigcup_{n=1}^{+\infty} \text{Ran}P_n = H.$$

Prove that (P_n) converges strongly to the identity operator as $n \to +\infty$. Show that (P_n) does not converge to the identity operator with respect to the operator norm unless $P_n = \mathbb{I}$ for all n sufficiently large.

15. Let $H = L^2(\mathbb{T}^3; \mathbb{R}^3)$ be the Hilbert space of 2π periodic, square-integrable, vector-valued functions $\mathbf{u} : \mathbb{T}^3 \to \mathbb{R}^3$, with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\mathbb{T}^3} \mathbf{u}(x) \cdot \mathbf{v}(x) dx.$$

We define the subspaces V and W of H by

$$V = \{\mathbf{v} \in C^\infty(\mathbb{T}^3; \mathbb{R}^3) \ : \ \text{div}\mathbf{v} = 0\},$$
$$W = \{\mathbf{w} \in C^\infty(\mathbb{T}^3; \mathbb{R}^3) \ : \ \mathbf{w} = \nabla\varphi, \text{ for some } \varphi : \mathbb{T}^3 :\to \mathbb{R}\}.$$

Show that H is the orthogonal direct sum of $\overline{V}$ and $\overline{W}$.

5.6 *Envisaged outcomes*

At the end of this chapter, the student should be familiar with

- The concepts of inner product space, Hilbert space, the main examples of Hilbert spaces, and simple properties such as the Cauchy-Schwarz inequality.
- The concepts of orthogonality, orthogonal projection, minimal distance from a closed subspace.
- The notion of orthonormal sequence, Bessel's inequality and its consequences, the notion of orthonormal basis.
- The notion of Fourier series in L^2. In the exercises, the student should be able to find the Fourier coefficients of a periodic function and discuss the convergence of the corresponding Fourier series.
- Know how to characterize the dual of a Hilbert space via the Riesz representation theorem.

6 Bounded operators on Hilbert spaces and introduction to spectral theory

In this chapter we describe some important classes of bounded linear operators on Hilbert spaces, including self-adjoint operators. We also introduce the Fredholm alternative principle and some properties of weak convergence on Hilbert spaces. Then, we introduce spectral theory for bounded linear operators on Hilbert spaces and derive some basic result such as the spectral decomposition of compact self-adjoint operators.

6.1 *The adjoint of an operator*

An important consequence of the Riesz representation theorem is the existence of the adjoint of a bounded operator on a Hilbert space. The defining property of the adjoint $A^* \in \mathcal{B}(H)$ of an operator $A \in \mathcal{M}(H)$ is that

$$\langle x, Ay \rangle = \langle A^* x, y \rangle \qquad \text{for all } x, y \in H.$$

To prove that A^* exists and is uniquely defined, we have to show that for every $x \in H$, there is a unique vector $z \in H$, depending linearly on x, such that

$$\langle z, y \rangle = \langle x, Ay \rangle \qquad \text{for all } x, y \in H. \tag{58}$$

For fixed x, the map φ_x defined by

$$\varphi_x(y) = \langle x, Ay \rangle$$

is a bounded linear functional on H, with $\|\varphi_x\| \le \|A\|\|x\|$. By the Riesz representation theorem, there is a unique $z \in H$ such that $\varphi_x(y) = \langle z, y \rangle$. This z satisfies (58), so we set $A^* x = z$. The linearity of A^* is left as an exercise.

6.1 example. The matrix of the adjoint of a linear map on $\mathbb{R}^n$ with matrix A is A^T, since

$$x \cdot (Ay) = (A^T x) \cdot y.$$

In component notation, we have

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} y_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_i \right) y_j.$$

The matrix of the adjoint of a linear map on $\mathbb{C}^n$ with complex matrix A is the Hermitian conjugate matrix,

$$A^* = \overline{A^T}.$$

6.2 example. Suppose that S and T are the right and left shift operators on the sequence space $\ell^2(\mathbb{N})$, defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots), \qquad T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Then $T = S^*$, since

$$\langle x, Sy \rangle = \overline{x_2} y_1 + \overline{x_2} y_2 + \overline{x_4} y_3 + \ldots = \langle Tx, y \rangle$$

6.3 exercise. Let $K : L^2([0,1]) \to L^2([0,1])$ be an integral operator of the form

$$Kf(x) = \int_0^1 k(x,y) f(y) dy,$$

where $k : [0,1] \times [0,1] \to \mathbb{C}$. Then prove that the adjoint operator

$$K^* f(x) = \int_0^1 \overline{k(y,x)} f(y) dy$$

is the integral operator with the complex conjugate, transpose kernel.

6.4 exercise. Prove that $A^{**} = A$ for all $A \in \mathcal{B}(H)$.

The adjoint plays a crucial role in studying the solvability of a linear equation

$$Ax = y \tag{59}$$

where $A : H \to H$ is a bounded linear operator. Let $z \in H$ be any solution of the homogeneous adjoint equation,

$$A^* z = 0.$$

We take the inner product of (59) with z. The inner product on the left-hand side vanishes because

$$\langle Ax, z \rangle = \langle x, A^* z \rangle = 0.$$

Hence, a necessary condition for a solution x of (59) to exist is that $\langle y, z \rangle = 0$ for all $z \in \ker A^*$, meaning that $y \in (\ker A^*)^\perp$. This condition on y is not always sufficient to guarantee the solvability of (59); the most we can say for general bounded operators is the following result. First, a simple exercise

6.5 exercise. Let $A \subset H$. Then $A^{\perp\perp} = (A^\perp)^\perp = \overline{[A]}$, where $\overline{[A]}$ is the closed linear span of A, i.e. the closure of the linear space generated by all finite linear combinations of vectors of A. To see this, we first observe that $A^{\perp\perp}$ is a closed subspace. Moreover, $A \subset A^{\perp\perp}$. Indeed, let $x \in A$, and let $y \in A^\perp$. Then $\langle x, y \rangle = 0$, which means $x \in A^{\perp\perp}$. So $A^{\perp\perp}$ is a closed linear subspace that contains A, and since $\overline{[A]}$ is the smallest linear subspace containing A we have

$A^{\perp\perp} \supset \overline{[A]}$. On the other hand, assuming that $\overline{[A]} \neq A^{\perp\perp}$, let $x \in A^{\perp\perp} \setminus \overline{[A]}$. We can always find $x \in \overline{[A]}^{\perp}$. Since $\overline{[A]} \supset A$, then $\overline{[A]}^{\perp} \subset A^{\perp}$, and therefore $x \in A^{\perp}$. But then $x \in A^{\perp} \cap A^{\perp\perp}$, which is only possible if $x = 0$. This proves the assertion.

6.6 THEOREM. *If $A : H \to H$ is a bounded linear operator, then*

$$\overline{\text{ran}A} = (\ker A^*)^{\perp}, \qquad \ker A = (\text{ran}A^*)^{\perp}. \tag{60}$$

Proof. If $x \in \text{ran}A$, there is a $y \in H$ such that $x = Ay$. For any $z \in \ker A^*$, we then havs

$$\langle x, z\rangle = \langle Ay, z\rangle = \langle y, A^*z\rangle = 0.$$

This proves that $\text{ran}A \subset (\ker A^*)^{\perp}$. Since $(\ker A^*)^{\perp}$ is closed, it follows that $\overline{\text{ran}A} \subset (\ker A^*)^{\perp}$. On the other hand, if $x \in (\text{ran}A)^{\perp}$, then for all $y \in H$ we have

$$0 = \langle Ay, x\rangle = \langle y, A^*x\rangle.$$

Since $y \in H$ is arbitrary, this implies that $A^*x = 0$, i.e. $x \in \ker A^*$. Hence, $(\text{ran}A)^{\perp} \subset \ker A^*$. By taking the orthogonal complement of this relation, we get

$$(\ker A^*)^{\perp} \subset (\text{ran}A)^{\perp\perp} = \overline{\text{ran}A},$$

which proves the first part of (60). To prove the second part, we apply the first part to A^* instead of A, use that the kernel of A is a closed linear subspace, use $A = A^{**}$, and take orthogonal complements. The details are left as an exercise. □

An equivalent formulation of this theorem is that if A is a bounded linear operator on H, then H is the orthogonal direct sum

$$H = \overline{\text{ran}A} \oplus \ker A^*.$$

If A has closed range, then we obtain the following necessary and sufficient condition for the solvability of (59)

6.7 THEOREM. *Suppose that $A : H \to H$ is a bounded linear operator on a Hilbert space H with closed range. Then the equation $Ax = y$ has a solution x if and only if y is orthogonal to* $\ker A^*$.

This theorem provides a useful general method of proving existence from uniqueness: if A has closed range, and the solution of the adjoint problem $A^*x = y$ is unique, then $\ker A^* = \{0\}$, so every y is orthogonal to $\ker A^*$. Hence, a solution of $Ax = y$ exists for every $y \in H$. The condition that A has closed range is implied by an estimate of the form $c\|x\| \leq \|Ax\|$, as shown in Proposition 4.46. A commonly occurring dichotomy for the solvability of a linear equation is summarized in the following *Fredholm alternative principle*.

6.8 definition. A bounded linear operator $A : H \to H$ on a Hilbert space H satisfies the *Fredholm alternative* if one of the following two alternatives holds:

(a) Either $Ax = 0$, $A^*x = 0$ have only the zero solution, and the equations $Ax = y$, $A^*x = y$ have a unique solution $x \in H$ for every $y \in H$,

(b) Or $Ax = 0$, $A^*x = 0$ have nontrivial, finite-dimensional solutions spaces of the same dimension, $Ax = y$ has a (nonunique) solution if and only if $y \perp z$ for every solution z of $A^*z = 0$, and $A^*x = y$ has a (nonunique) solution if and only if $y \perp z$ for every solution z of $Az = 0$.

Any linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ on a finite-dimensional space, associated with an $n \times n$ system of linear equations $Ax = y$, satisfies the Fredholm alternative. The ranges of A and A^* are closed because they are finite-dimensional. From linear algebra, the rank of A^* is equal to the rank of A, and therefore the nullity of A is equal to the nullity of A^*. The Fredholm alternative then follows from Theorem 6.7. Two things can go wrong with the Fredholm alternative in Definition 6.8 for bounded operators A on an infinite-dimensional space. First, ran$A$ need not be closed; and second, even if ran$A$ is closed, it is not true, in general, that ker$A$ and kerA^* have the same dimension. As a result, the equation $Ax = y$ may be solvable for all $y \in H$ even though A is not one-to-one, or $Ax = y$ may not be solvable for all $y \in H$ even though A is one-to-one. We illustrate these possibilities with some examples.

6.9 example. Consider the multiplication operator $M : L^2([0, 1]) \to L^2([0, 1])$ defined by

$$Mf(x) = xf(x).$$

Then $M^* = M$, and M is one-to-one, so every $g \in L^2([0, 1])$ is orthogonal to kerM^*; but the range of M is a proper dense subspace of $L^2([0, 1])$, so $Mf = g$ is not solvable for every $g \in L^2([0, 1])$. We will get back to this example below.

6.10 example. The range of the right shift operator $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined in Example 6.2, is closed since it consists of $y = (y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N})$ such that $y_1 = 0$. The left shift operator $T = S^*$ is singular since its kernel is the one-dimensional space with basis $\{(1, 0, 0, \ldots)\}$. The equation $Sx = y$, or $(0, x_1, x_2, \ldots) = (y_1, y_2, y_3, \ldots)$, is solvable if and only if $y_1 = 0$, or $y \perp$ ker$T$, which verifies Theorem 6.7 in this case. If a solution exists, then it is unique. On the other hand, the equation $Tx = y$ is solvable for every $y \in \ell^2(\mathbb{N})$, even though T is not one-to-one, and the solution is not unique.

6.11 definition. A bounded linear operator $A : H \to H$ on a Hilbert space is *self-adjoint* if $A^* = A$.

Equivalently, a bounded linear operator A on H is self-adjoint if

$$\langle x, Ay\rangle = \langle Ax, y\rangle \qquad \text{for all } x, y \in H.$$

6.12 example. From Example 6.1, a linear map on $\mathbb{R}^n$ with matrix A is self-adjoint if and only if A is symmetric, meaning that $A = A^T$, where A^T is the transpose of A. A linear map on $\mathbb{C}^n$ with matrix A is self-adjoint if and only if A is *Hermitian*, meaning that $A = A^*$.

6.13 example. From example 6.3, an integral operator $K : L^2([0,1]) \to L^2([0,1])$

$$Kf(x) = \int_0^1 k(x,y) f(y) dy,$$

is self-adjoint if and only if $k(x,y) = \overline{k(y,x)}$.

Given a linear operator $A : H \to H$, we may define a sesquilinear form

$$a : H \times H \to \mathbb{C}$$

by $a(x,y) = \langle x, Ay \rangle$. If A is self-adjoint, then this form is *Hermitian symmetric*, or symmetric, meaning that

$$a(x,y) = \overline{a(y,x)}.$$

It follows that the associated quadratic form $q(x) = a(x,x)$, or

$$q(x) = \langle x, Ax \rangle,$$

is real-valued. We say that A is *nonnegative* if it is self-adjoint and $\langle x, Ax \rangle \geq 0$ for all $x \in H$. We say that A is *positive*, or *positive definite*, if it is self-adjoint and $\langle x, Ax \rangle > 0$ for every nonzero $x \in H$. If A is a positive, bounded operator, then

$$(x,y) = \langle x, Ay \rangle$$

defines an inner product on H. If, in addition, there is a constant $c > 0$ such that

$$\langle x, Ax \rangle \geq c\|x\|^2 \qquad \text{for all } x \in H,$$

then we say that A is *bounded from below*, and the norm associated with $(\cdot,\cdot)$ is equivalent to the norm associated with $\langle \cdot, \cdot \rangle$.

The quadratic form associated with a self-adjoint operator determines the norm of the operator.

6.14 lemma (Norm of an adjoint operator via quadratic form). *If A is a bounded self-adjoint operator on a Hilbert space H, then*

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle|.$$

Proof. Let

$$\alpha = \sup_{\|x\|=1} |\langle x, Ax \rangle|.$$

The inequality $\alpha \leq \|A\|$ is immediate, since

$$|\langle x, Ax \rangle| \leq \|Ax\|\|x\| \leq \|A\|\|x\|^2.$$

To prove the reverse inequality, we use the definition of the norm,

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

For any $z \in H$ we have

$$\|z\| = \sup_{\|y\|=1} |\langle y, z \rangle|.$$

It follows that

$$\|A\| = \sup\{|\langle y, Ax \rangle| \, : \, \|x\| = 1, \, \|y\| = 1\}. \tag{61}$$

A tedious but simple computation yields

$$\langle y, Ax \rangle = \frac{1}{4} \left[\langle x+y, A(x+y) \rangle - \langle x-y, A(x-y) \rangle \right.$$
$$\left. -i\langle x+iy, A(x+iy) \rangle + i\langle x-iy, A(x-iy) \rangle \right].$$

Since A is self-adjoint, the first two terms above are real, and the last two are imaginary. We replace y by $e^{i\varphi}y$ where $\varphi \in \mathbb{R}$ is chosen so that $\langle e^{i\varphi}y, Ax \rangle$ is real. Then the imaginary terms vanish, and we find that

$$\begin{aligned}
|\langle y, Ax \rangle|^2 &= \frac{1}{16} \left| \langle x+y, A(x+y) \rangle - \langle x-y, A(x-y) \rangle \right|^2 \\
&\leq \frac{1}{16}\alpha^2 (\|x+y\|^2 + \|x-y\|^2)^2 \\
&= \frac{1}{4}\alpha^2 (\|x\|^2 + \|y\|^2)^2,
\end{aligned}$$

where we have used the definition of α and the parallelogram law. Using this result in (61), we conclude that $\|A\| \leq \alpha$. □

As a corollary, we have the following result.

6.15 corollary. *If A is a bounded operator on a Hilbert space then $\|AA^*\| = \|A\|^2$. If A is self adjoint, then $\|A^2\| = \|A\|^2$.*

Proof. The definition of $\|A\|$ and the application of Lemma 6.14 to the self-adjoint operator A^*A imply that

$$\|A\|^2 = \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \sup_{\|x\|=1} |\langle x, A^*Ax\rangle| = \|A^*A\|.$$

Hence, if A is self-adjoint, then $\|A\|^2 = \|A^2\|$. □

6.2 *Weak convergence in a Hilbert space*

A sequence in a Hilbert space H converges weakly to $x \in H$ if

$$\lim_{n\to+\infty} \langle x_n, y\rangle = \langle x, y\rangle \qquad \text{for all } y \in H.$$

Weak convergence is usually written as

$$x_n \rightharpoonup x \qquad \text{as } n \to +\infty,$$

to distinguish it from strong, or norm, convergence.

6.16 example. Suppose that $H = \ell^2(\mathbb{N})$. Let

$$e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$$

be the standard basis vector whose n-th term is 1 and whose other terms are 0. If $(y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N})$, then

$$\langle e_n, y\rangle = y_n \to 0 \qquad \text{as } n \to +\infty,$$

since $\sum |y_n|^2$ converges. Hence, $e_n \rightharpoonup 0$ as $n \to +\infty$. On the other hand, $\|e_n - e_m\| = \sqrt{2}$ for all $n \neq m$, so the sequence (e_n) cannot converge strongly.

Clearly, all the result we have proven on the weak topologies of Banach spaces apply to Hilbert spaces.

6.17 example. In example 6.16, we saw that the bounded sequence (e_n) of standard basis elements in $\ell^2(\mathbb{N})$ converges weakly to zero. The unbounded sequence (ne_n), where

$$ne_n = (0, 0, \ldots, 0, n, 0, \ldots),$$

does not converge weakly, however, even though the coordinate sequence with respect to the basis (e_n) converges to zero. For example,

$$x = \left(n^{-3/4}\right)_{n=1}^{+\infty}$$

belongs to $\ell^2(\mathbb{N})$, and $\langle ne_n, x\rangle = n^{1/4}$ does not converge as $n \to +\infty$.

The examples we saw in subsection 3.4 on the weak convergence in L^p spaces in the case $p = 2$ are a special case of weak convergence in a Hilbert space. In particular, the phenomena of oscillation, concentration, and escape to infinity can occur.

As we know, the norm of the limit of a weakly convergent sequence may be strictly less than the norms of the terms in the sequence, corresponding to a loss of 'energy' in oscillations, at a singularity, or by escape to infinity in the weak limit. In each case, the expansion of f_n in any orthonormal basis contains coefficients that wander off to infinity. If the norms of a weakly convergent sequence converge to the norm of the weak limit, then the sequence converges strongly.

6.18 PROPOSITION. *If* (x_n) *converges weakly to* x *and*

$$\lim_{n\to+\infty} \|x_n\| = \|x\|,$$

then (x_n) *converges strongly to* x.

Proof. Expansion of the inner product gives

$$\|x_n - x\|^2 = \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2.$$

If x_n converges weakly to x, then $\langle x_n, x\rangle \to < \langle x, x\rangle = \|x\|^2$. Hence, if we also have $\|x_n\| \to \|x\|$, then $\|x_n - x\|^2 \to 0$, meaning that $x_n \to x$ strongly. $\square$

6.19 DEFINITION. We say that a functional $f : X \to \mathbb{R}$ defined on a Banach space X is *weakly lower semicontinuous* if for every sequence (x_n) in X which converges weakly to $x \in X$, we have

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$

We say that a subset $C \subset X$ of a Banach space is *weakly closed* if for every sequence x_n in C which converges weakly to $x \in X$ we have $x \in C$.

6.20 THEOREM (Direct method of calculus of variations). *Suppose that* $f : K \to \mathbb{R}$ *is a weakly lower semicontinuous functional on a weakly closed, bounded subset* K *of a Hilbert space* H. *Then* f *is bounded from below and attains its infimum, i.e. there exists* $x \in K$ *such that*

$$f(x) = \min_{y\in K} f(y).$$

Proof. Let $m = \inf_{y\in K} f(y)$. Let y_n be a sequence in K such that $f(y_n) \to m$. Since K is bounded in a reflexive Banach space, there is a subsequence y_{n_k} of y_n which converges weakly to some $y \in H$. Since K is weakly closed, we have that $y \in K$. Since f is weakly lower semicontinuous, we have

$$f(y) \leq \liminf_{k\to+\infty} f(y_{n_k}) = m,$$

hence $f(y) \leq d$. But m is the infimum of f on K, and $y \in K$. Therefore $f(y) = m$, which proves the assertion. $\square$

The above theorem shows the tremendous impact of weak compactness theorems on the calculus of variation, see example 0.8. Now we have the technology to achieve the solution of a minimization problem on an infinite dimensional space. Clearly, the price we pay in order to get existence of a minimum is that the functional should be weakly lower semicontinuous and the set K be weakly closed. Such assumptions are clearly stricter than strong lower semicontinuity and strong closedness, since there are more weakly convergent sequences than strongly convergent ones, so one has 'more tests to perform' in order to check both conditions.

6.3 *The spectrum*

Spectral theory provides a powerful way to understand linear operators by decomposing the space on which they act into invariant subspaces on which their action is simple. In the finite-dimensional case, the spectrum of a linear operator consists of its eigenvalues. The action of the operator on the subspace of eigenvectors with a given eigenvalue is just multiplication by the eigenvalue. As we will see, the spectral theory of bounded linear operators on infinite dimensional spaces is more involved. For example, an operator may have a *continuous spectrum* in addition to, or instead of, a *point spectrum* of eigenvalues. A particularly simple and important case is that of compact, self-adjoint operators. Compact operators may be approximated by finite-dimensional operators, and their spectral theory is close to that of finite-dimensional operators.

The student at this level should be familiar with the diagonalization of squared matrices, with concepts such as eigenvalue, eigenvector, eigenspace, algebraic multiplicity and geometric multiplicity of an eigenvalue, characteristic polynomial of a matrix. The student should also be familiar with the following important result in linear algebra: every self-adjoint squared (finite dimensional) matrix has an orthonormal basis of eigenvectors (which means it is diagonalisable).

A bounded linear operator on an infinite-dimensional Hilbert space need not have any eigenvalues at all, even it if is self-adjoint. Thus, we cannot hope to find, in general, an orthonormal basis of the space consisting entirely of eigenvectors. It is therefore necessary to define the spectrum of a linear operator on an infinite-dimensional space in a more general way than as the set of eigenvalues. We denote the space of bounded linear operators on a Hilbert space H by $\mathcal{L}(H)$.

6.21 definition. The *resolvent set* of an operator $A \in \mathcal{L}(H)$, denoted by $\rho(A)$, is the set of complex numbers λ such that $(A - \lambda \mathbb{I}) : H \to H$ is one-to-one and onto. The *spectrum* of A, denoted by $\sigma(A)$, is the complement of the resolvent set in $\mathbb{C}$, meaning that $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

If $A - \lambda\mathbb{I}$ is one-to-one and onto, then the open mapping theorem implies that $(A - \lambda\mathbb{I})^{-1}$ is bounded. Hence, when $\lambda \in \rho(A)$, both $A - \lambda\mathbb{I}$ and $(A - \lambda\mathbb{I})^{-1}$ are one-to-one, onto, bounded linear operators.

As in the finite-dimensional case, a complex number λ is called an eigenvalues of A if there is a nonzero vector $u \in H$ such that $Au = \lambda u$. In that case,

$\ker(A - \lambda\mathbb{I}) \neq \{0\}$, so $A - \lambda\mathbb{I}$ is not one-to-one, and $\lambda \in \sigma(A)$. This is not the only way, however, that a complex number can belong to the spectrum. We subdivide the spectrum of a bounded linear operator as follows.

6.22 DEFINITION. Suppose that A is a bounded linear operator on a Hilbert space H.

(a) The *point spectrum* of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda\mathbb{I}$ is not one-to-one. In this case λ is called an *eigenvalue*s of A.

(b) The *continuous spectrum* of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda\mathbb{I}$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda\mathbb{I})$ is dense in H.

(c) The *residual spectrum* of A consists of all $\lambda \in \sigma(A)$ such that $A - \lambda\mathbb{I}$ is one-to-one but not onto, and $\operatorname{ran}(A - \lambda\mathbb{I})$ is not dense in H.

6.23 EXAMPLE. Let $H = L^2([0,1])$, and define the multiplication operator $M : H \to H$ by

$$Mf(x) = xf(x).$$

Then M is bounded with $\|M\| = 1$ (exercise!). If $Mf = \lambda f$, then it is easily seen that $f(x) = 0$ for almost every $x \in [0,1]$, so $f = 0$ in $L^2([0,1])$. Thus, f has no eigenvalues. If $\lambda \notin [0,1]$, then $(x-\lambda)^{-1}f(x) \in L^2([0,1])$ for any $f \in L^2([0,1])$ because $(x - \lambda)$ is bounded away from zero on $[0,1]$. Thus, $\mathbb{C} \setminus [0,1]$ is in the resolvent set of M. If $\lambda \in [0,1]$, then $M - \lambda\mathbb{I}$ is not onto, because $c(x-\lambda)^{-1} \notin L^2([0,1])$ for $c \neq 0$, so the nonzero constant function c does not belong to the range of $M - \lambda\mathbb{I}$. The range of $M - \lambda\mathbb{I}$, however, is dense. To see this, let $f \in L^2([0,1])$, let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x-\lambda| \geq 1/n \\ 0 & \text{if } |x - \lambda| < 1/n. \end{cases}$$

Then f_n converges to f in $L^2([0,1])$, and $f_n \in \operatorname{Ran}(M - \lambda\mathbb{I})$, since $(x - \lambda)^{-1}f_n(x) \in L^2([0,1])$. It follows that $\sigma(M) = [0,1]$, and that every $\lambda \in \sigma(M)$ belongs to the continuous spectrum of M.

If λ belongs to the resolvent set $\rho(A)$ of a linear operator A, then $A - \lambda\mathbb{I}$ has an everywhere defined, bounded inverse. The operator

$$R_\lambda = (\lambda\mathbb{I} - A)^{-1}$$

is called the *resolvent operator* of A at λ. The resolvent operator of A is an operator-valued function defined on the subset $\rho(\mathbb{C})$.

An operator-valued function $F : \Omega \to \mathcal{L}(H)$, defined on an open subset Ω of the complex plane $\mathbb{C}$ is said to be *analytic* at $z_0 \in \Omega$ if there are operators $F_n \in \mathcal{L}(H)$ and a $\delta > 0$ such that

$$F(x) = \sum_{n=0}^{+\infty} (z - z_0)^n F_n,$$

where the power series on the right-hand side converges with respect to the operator norm on $\mathcal{L}(H)$ in a disc $|z - z_0| < \delta$. We say that F is analytic in Ω if it is analytic at any point of Ω.

6.24 exercise (Neumann Series). Suppose that $K : X \to X$ is a bounded linear operator on a Banach space X with $\|K\| < 1$. Prove that $\mathbb{I} - K$ is invertible and

$$(\mathbb{I} - K)^{-1} = \mathbb{I} + K + K^2 + K^3 + \ldots,$$

where the series on the right hand side converges uniformly in $\mathcal{L}(X)$.

6.25 proposition. *If A is a bounded linear operator on a Hilbert space H, then the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ that contains the exterior disc $\{\lambda \in \mathbb{C} : |\lambda| > \|A\|\}$. The resolvent R_λ is an operator valued analytic function of λ on $\rho(A)$.*

Proof. Suppose that $\lambda_0 \in \rho(A)$. Then we may write

$$\lambda\mathbb{I} - A = (\lambda_0\mathbb{I} - A)[\mathbb{I} - (\lambda_0 - \lambda)(\lambda_0\mathbb{I} - A)^{-1}].$$

If $|\lambda_0 - \lambda| < \|(\lambda_0\mathbb{I} - A)^{-1}\|^{-1}$, then we can invert the operator on the right-hand-side by the Neumann series (see exercise 6.24). Hence, there is an open disk in the complex plane with center λ_0 that is contained in $\rho(A)$. Moreover, the resolvent R_λ is given by an operator-norm convergent Taylor series in the disc, so it is analytic in $\rho(A)$. To see this, we compute

$$R_\lambda = [\mathbb{I} - (\lambda_0 - \lambda)R_{\lambda_0}]^{-1}R_{\lambda_0} = \sum_{k=1}^{+\infty}(\lambda_0 - \lambda)^k R_{\lambda_0}^{k+1}.$$

If $|\lambda| > \|A\|$, then the Neumann series also shows that $\lambda(\mathbb{I} - A\lambda^{-1})$ is invertible, so $\lambda \in \rho(A)$. □

Since the spectrum $\sigma(A)$ of A is the complement of the resolvent set, it follows that the spectrum is a closed subset of $\mathbb{C}$, and

$$\sigma(A) \subset \{z \in \mathbb{C} : |z| \leq \|A\|\}.$$

The *spectral radius* of A, denoted by $r(A)$, is the radius of the smallest disk centered at zero that contains $\sigma(A)$,

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

We can refine the above proposition 6.25 as follows.

6.26 proposition. *If A is a bounded linear operator, then*

$$r(A) = \lim_{n\to+\infty} \|A^n\|^{1/n}. \tag{62}$$

If A is self-adjoint, then $r(A) = \|A\|$.

Proof. Omitted.

Although the spectral radius of a self-adjoint operator is equal to its norm, the spectral radius does not provide a norm on the space of all bounded operators. In particular, $r(A) = 0$ does not imply that $A = 0$. If $r(A) = 0$, then we say that A is a *nilpotent operator* (as an example consider a nontrivial Jordan block).

6.27 PROPOSITION. *The spectrum of a bounded linear operator on a Hilbert space is nonempty.*

We omit the proof.

6.4 *The spectral theorem for compact, self-adjoint operators*

In this section, we analyze the spectrum of a compact, self-adjoint operator. The spectrum consists entirely of eigenvalues, with the possible exception of zero, which may belong to the continuous spectrum. We begin by proving some basic properties of the spectrum of a bounded, self-adjoint operator.

6.28 LEMMA. *The eigenvalues of a bounded, self-adjoint operator are real, and eigenvectors associated with different eigenvalues are orthogonal.*

Proof. If $A : H \to H$ is self-adjoint, and $Ax = \lambda x$ with $x \neq 0$, then

$$\lambda\langle x, x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle = \overline{\lambda}\langle x, x\rangle,$$

and $\overline{\lambda} = \lambda$, i.e. $\lambda \in \mathbb{R}$. If $Ax = \lambda x$ and $Ay = \mu y$, where λ and μ are real, then

$$\lambda\langle x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \mu\langle x, y\rangle.$$

It follows that if $\lambda \neq \mu$, then $\langle x, y\rangle = 0$ and $x \perp y$. □

A linear subspace M of H is called an *invariant subspace* of a linear operator A on H if $Ax \in M$ for all $x \in M$. In that case, the restriction $A|_M$ of A to M is a linear operator on M. Suppose that $H = M \oplus N$ is the direct sum of invariant subspaces M and N of A. Then every $x \in H$ may be written as $x = y + z$ with $y \in M$ and $z \in N$, and

$$Ax = A|_M y + A|_N z.$$

Thus, the action of A on H is determined by the actions on the invariant subspaces.

6.29 EXAMPLE. Consider matrices acting on $\mathbb{C}^d = \mathbb{C}^m \oplus \mathbb{C}^n$, where $d = m + n$. A $d \times d$ matrix A leaves $\mathbb{C}^m$ invariant if it has the block form

$$A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix},$$

where B is an $m \times m$ matrix, D is $m \times n$, and C is $n \times n$. The matrix A leaves both $\mathbb{C}^m$ and the complementary subspace $\mathbb{C}^n$ invariant if $D = 0$.

An invariant subspace of a non-diagonalizable operator may have no complementary invariant subspace. However, the orthogonal complement of an invariant subspace of a self-adjoint operator is also invariant, as we prove in the following lemma. Thus, we can decompose the action of a self-adjoint operator on a linear space into action on smaller orthogonal invariant subspaces.

6.30 lemma. *If A is a bounded, self-adjoint operator on a Hilbert space H and M is an invariant subspace of A, then $M^\perp$ is an invariant subspace of A.*

Proof. If $x \in M^\perp$ and $y \in M$, then

$$\langle y, Ax\rangle = \langle Ay, x\rangle = 0$$

because $A = A^*$ and $Ay \in M$. Therefore, $Ax \in M^\perp$. □

Next we show that the whole spectrum - not just the point spectrum - of a bounded, self-adjoint operator is real, and that the residual spectrum is empty. We begin with a preliminary proposition.

6.31 proposition. *If λ belongs to the residual spectrum of a bounded operator A on a Hilbert space, then $\overline{\lambda}$ is an eigenvalues of A^*.*

Proof. If λ belongs to the residual spectrum of a bounded operator $A \in \mathcal{L}(H)$, then $\operatorname{Ran}(A - \lambda\mathbb{I})$ is not dense in H. Hence, there is a nonzero vector $x \in H$ such that $x \perp \operatorname{Ran}(A - \lambda\mathbb{I})$. Theorem 6.6 then implies that $x \in \operatorname{Ker}(A^* - \overline{\lambda}\mathbb{I})$. Hence, $\overline{\lambda}$ is an eigenvalue of A^*. □

6.32 lemma. *If A is a bounded, self-adjoint operator on a Hilbert space, then the spectrum of A is real and is contained in the interval $[-\|A\|, \|A\|]$.*

Proof. We have shown that $r(A) \leq \|A\|$, so we only have to prove that the spectrum is real. Suppose that $\lambda = a + ib \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and $b \neq 0$. For any $x \in H$, we have

$$\begin{aligned}
\|(A - \lambda\mathbb{I})x\|^2 &= \langle (A - \lambda\mathbb{I})x, (A - \lambda\mathbb{I})x\rangle \\
&= \langle (A - a\mathbb{I})x, (A - a\mathbb{I})x\rangle + \langle (-ib)x, , (-ib)x\rangle \\
&\quad + \langle (A - a\mathbb{I})x, (-ib)x\rangle + \langle (-ib)x, (A - a\mathbb{I})x\rangle \\
&= \|(A - a\mathbb{I})x\|^2 + b^2\|x\|^2 \geq b^2\|x\|^2.
\end{aligned}$$

It follows from this estimate and Proposition 4.46 that $A - \lambda\mathbb{I}$ is one-to-one and has closed range. If $\operatorname{Ran}(A - \lambda\mathbb{I}) \neq H$, then λ belongs to the residual spectrum of A, and, by proposition 6.31, $\overline{\lambda} = a - ib$ is an eigenvalue of A. Thus, A has an eigenvalue that does not belong to $\mathbb{R}$, which contradicts Lemma 6.28. It follows that $\lambda \in \rho(A)$ if λ is not real. □

6.33 corollary. *The residual spectrum of a bounded, self-adjoint operator is empty.*

Proof. From Lemma 6.32, the point spectrum and the residual spectrum are disjoint subsets of $\mathbb{R}$. So if $\lambda \in \mathbb{R}$ belongs to the residual spectrum, Proposition

6.31 implies that λ is also an eigenvalue, which is a contradiction. □

Bounded linear operators on an infinite dimensional Hilbert space do not always behave like operators on a finite dimensional space. We have seen in Example 6.23 that a bounded, self-adjoint operator may have no eigenvalues, while the identity operator on an infinite dimensional Hilbert space has a nonzero eigenvalues of infinite multiplicity. The properties of compact operators are much closer to those of operators on finite dimensional spaces, and we will study their spectral theory next.

6.34 proposition. *A nonzero eigenvalue of a compact, self-adjoint operator A has finite multiplicity. A countably infinite set of nonzero eigenvalues has zero as accumulation point, and no other accumulation points.*

Proof. Suppose by contradiction that λ is a nonzero eigenvalue with infinite multiplicity. Then, there is a sequence (e_n) of orthonormal eigenvectors. This sequence is bounded, but (Ae_n) does not have a convergent subsequence because $Ae_n = \lambda e_n$. Indeed, $\|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 1$. This contradicts the compactness of A.

If A has a countably infinite set $\{\lambda_n\}$ of nonzero eigenvalues, with e_n as corresponding orthonormal eigenvectors, then, since the eigenvalues are bounded by $\|A\|$, there is a convergent subsequence (λ_{n_k}). If $\lambda_{n_k} \to \lambda$ and $\lambda \neq 0$, then the orthogonal sequence of eigenvectors (f_{n_k}), where $f_{n_k} = \lambda_{n_k}^{-1} e_{n_k}$ and $\|e_{n_k}\| = 1$, would be bounded. But (Af_{n_k}) has no convergent subsequence since $Af_{n_k} = e_{n_k}$. □

To motivate the statement of the spectral theorem for compact, self-adjoint operators, suppose that $x \in H$ is given by

$$x = \sum_k c_k e_k + z, \tag{63}$$

where $\{e_k\}$ is an orthonormal set of eigenvectors of A with corresponding nonzero eigenvalues λ_k, $z \in \ker(A)$, and $c_k \in \mathbb{C}$. Then

$$Ax = \sum_k \lambda c_k e_k.$$

Let P_k denote the one-dimensional orthogonal projection onto the subspace spanned by e_k,

$$P_k x = \langle e_k, x \rangle e_k.$$

From Lemma 6.28, we have $z \perp e_k$, so $c_k = \langle e_k, x \rangle$ and

$$Ax = \sum_k \lambda_k P_k x. \tag{64}$$

If λ_k has finite multiplicity $m_k > 1$, meaning that the dimension of the associated eigenspace $E_k \subset H$ is greater than one, then we may combine the one-dimensional projections associated with the same eigenvalues. In doing so, we may represent A by a sum of the same form as (64), in which λ_k are distinct, and the P_k are orthogonal projections onto the eigenspaces E_k.

The spectral theorem for compact, self-adjoint operators states that any $x \in H$ can be expanded in the form (63), and that A can be expressed as a sum of orthogonal projections as in (64).

6.35 theorem (Spectral theorem for compact, self-adjoint operators). *Let $A : H \to H$ be a compact, self-adjoint operator on a Hilbert space H. There is an orthonormal basis of H consisting of eigenvectors of A. The nonzero eigenvalues of A form a finite or countably infinite set $\{\lambda_k\}$ of real numbers, and*

$$A = \sum_k \lambda_k P_k, \tag{65}$$

where P_k is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalues λ_k. If the number of nonzero eigenvalues is countably infinite, then the series in (65) *converges to A in the operator norm.*

We omit the proof.

The above theorem is only one simple version of spectral theorem for compact operators. It can be generalized and extended to many other cases which will not be considered here. We emphasize in particular that the property

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$$

holds for *all compact operators on a Hilbert space*, not only for the self-adjoint ones. We omit the proof.

6.5 *More on compact operators*

Before we can apply the spectral theorem for compact, self-adjoint operators, we have to check that an operator is compact. In this subsection we discuss some ways to do this, and also give some examples of compact operators.

The most direct way to prove that an operator is compact is to verify the definition by showing that if E is a bounded subset of H, then the set $A(E) = \{Ax : x \in E\}$ is precompact, i.e. with compact closure. In many examples, this can be done by using an appropriate condition for compactness, such as the Arzelá-Ascoli theorem or Kolmogorov-Riesz-Frechet Theorem. The following theorem characterizes precompact sets in a general, separable Hilbert space. We omit the proof.

6.36 theorem. *Let E be a subset of an infinite-dimensional, separable Hilbert space H.*

(a) If E is precompact, then for every orthonormal set $\{e_n : n \in \mathbb{N}\}$ and every $\epsilon > 0$, there is an N such that

$$\sum_{n=N+1}^{+\infty} |\langle e_n, x\rangle|^2 < \epsilon \qquad \textit{for all } x \in E. \tag{66}$$

(b) *If E is bounded and there is an orthonormal basis $\{e_n\}$ of H with the property that for every $\epsilon > 0$ there is an N such that* (66) *holds, then E is precompact.*

6.37 example. Let $H = \ell^2(\mathbb{N})$. The *Hilbert cube*

$$C = \{(x_1, x_2, x_3, \ldots, x_n, \ldots) \ : \ |x_n| \leq 1/n\}$$

is closed and precompact. Hence C is a compact subset of H.

6.38 exercise. Consider the diagonal operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_2 x_3, \ldots, \lambda_n x_n, \ldots), \tag{67}$$

where $\lambda_n \in \mathbb{R}$ and λ_n is decreasing with $\lambda_n \to 0$ as $n \to +\infty$. Prove that A is compact.

Proposition 4.26 implies that the uniform limit of compact operators is compact. An operator with finite rank is compact. Therefore, another way to prove that A is compact is to show that A is the limit of a uniformly convergent sequence of finite-rank operators. One such class of compact operators is the class of Hilbert-Schmidt operators.

6.39 definition. A bounded linear operator A on a separable Hilbert space H is *Hilbert-Schmidt* if there is an orthonormal basis $\{e_n \ : \ n \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{+\infty} \|Ae_n\|^2 < +\infty \tag{68}$$

If A is Hilbert-Schmidt, then

$$\|A\|_{HS} = \sqrt{\sum_{n=1}^{+\infty} \|Ae_n\|^2} \tag{69}$$

is called the *Hilbert-Schmidt norm* of A.

One can show that the sum in (68) is finite in every orthonormal basis if it is finite in one orthonormal basis, and the norm (69) does not depend on the choice of the basis.

6.40 theorem. *A Hilbert-Schmidt operator is compact.*

Proof. (Sketch) Suppose that A is a Hilbert-Schmidt operator and $\{e_n \ : \ n \in \mathbb{N}\}$ is an orthonormal basis. If P_N is the orthogonal projection onto the finite-dimensional space spanned by $\{e_1, \ldots, e_n\}$, then $P_n A$ is a finite rank operator, and one can check that $P_N A \to A$ uniformly as $N \to +\infty$. ☐

6.41 example. The diagonal operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined in (67) is

Hilbert-Schmidt if and only if

$$\sum_{n=1}^{+\infty} |\lambda_n|^2 < +\infty.$$

6.42 example. Let $\Omega \subset \mathbb{R}^n$. One can show that an integral operator K on $L^2(\Omega)$,

$$Kf(x) = \int_\Omega k(x,y)f(y)dy,$$

is Hilbert-Schmidt if and only if $k \in L^2(\Omega \times \Omega)$, meaning that

$$\int_{\Omega\times\Omega} |k(x,y)|^2 dxdy < +\infty.$$

The Hilbert-Schmidt norm of K is

$$\|K\|_{HS} = \left(\int_{\Omega\times\Omega} |k(x,y)|^2 dxdy \right)^{1/2}.$$

If K is a self-adjoint, Hilbert-Schmidt operator then there is an orthonormal basis $\{\varphi_n \,:\, n \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenvectors of K, such that

$$\int_\Omega k(x,y)\varphi_n(y)dy = \lambda_n \varphi_n(y).$$

Then, one can prove that

$$k(x,y) = \sum_{n=1}^{+\infty} \lambda_n \varphi_n(x)\varphi_n(y).$$

We omit the proof.

6.43 exercise. Prove that the Volterra operator

$$(Tf)(x) = \int_0^x f(t)dt$$

is compact in $H = L^2([0,1])$.

Solution. Let $B_1(0)$ be the closed unit ball of H. For $f \in B_1(0)$ we consider

$$\int_0^1 |(Tf)(x+h) - (Tf)(x)|^2 \, dx = \int_0^1 \left| \int_x^{x+h} f(t)dt \right|^2 dx$$
$$\leq \int_0^1 \left| \int_x^{x+h} dt \right| \left| \int_x^{x+h} f(t)^2 dt \right| dx \leq \int_0^1 |h| \int_0^1 f(t)^2 dt dx \leq |h| \, .$$

Hence,

$$\|(Tf)(\cdot + h) - (Tf)\|_{L^2} \leq |h|^{1/2} \, ,$$

and the assertion follows form Kolmogorov-Riesz-Frechet Theorem.

6.44 exercise. Prove that the Volterra operator in the exercise above has no eigenvalues.
Solution. Assume $\lambda \neq 0$ is an eigenvalue. Then, for $f \neq 0$ in L^2 we have

$$\lambda f(x) = \int_0^x f(t)dt \, .$$

For $x, y \in [0, 1]$,

$$f(x) - f(y) = \frac{1}{\lambda} \int_y^x f(t)dt$$

and therefore

$$|f(x) - f(y)| \leq \frac{|x-y|^{1/2}}{\lambda} \|f\|_{L^2}$$

which shows that f is continuous. Hence, the eigenvalue condition implies f is C^1. We can therefore write

$$\lambda f'(x) = f(x)$$

and since $f(0) = 0$ we get, by uniqueness of solutions to the Cauchy problem, that $f = 0$, a contradiction.

6.6 *Exercises*

1. Let H be a Hilbert space. Prove that for all $A, B \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$, one has
   (a) $A^{**} = A$
   (b) $(AB)^* = B^* A^*$
   (c) $(\lambda A)^* = \overline{\lambda} A^*$

(d) $(A+B)^* = A^* + B^*$

(e) $\|A^*\| = \|A\|$.

2. Let (u_n) be a sequence of orthonormal vectors in a Hilbert space. Prove that u_n converges to zero weakly.

3. Let $A \in \mathcal{L}(H)$ with H a Hilbert space. Prove that $\rho(A^*) = \overline{\rho(A)}$.

4. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$k(x,y) = \sum_{i=1}^{N} \alpha_i(x)\beta_i(y),$$

for some continuous functions $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$ on $[0,1]$. Prove that the operator $K \in \mathcal{L}(L^2([0,1]))$ defined by

$$(Kf)(x) = \int_0^1 k(x,y)f(y)dy$$

is compact.

5. Let λ be an eigenvalue of $A \in \mathcal{L}(H)$ with H a Hilbert space. Is $\overline{\lambda}$ in the spectrum of A^*? What can we say about the type of spectrum $\overline{\lambda}$ belongs to?

6. Suppose $A \in \mathcal{L}(H)$ with H a Hilbert space and $\lambda, \mu \in \rho(A)$. Prove that the resolvent R_λ of A satisfies the *resolvent equation*

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu.$$

7. Let H be a Hilbert space and $M \subset H$ be a closed subspace. Let $P : H \to H$ be the orthogonal projection of H onto M. Find $\sigma(M)$.

8. Let A be a bounded, self-adjoint, nonnegative operator on a complex Hilbert space. Prove that $\sigma(A) \subset [0, +\infty)$.

9. Let G be a multiplication operator on $L^2([0,1])$ defined by

$$Gf(x) = x^2 f(x).$$

Prove that G is a bounded linear operator on $L^2([0,1])$ and that its spectrum is given by $[0,1]$. Does G have eigenvalues? Motivate your answer.

10. Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)dy.$$

(a) Find the adjoint operator K^*

(b) Find the operator norm $\|K\|$ (Hint: use $\|K^*K\| = \|K\|^2$).

(c) Show that the spectral radius of K is equal to zero.

(d) Show that 0 belongs to the continuum spectrum of K.

11. Let $\ell^2(\mathbb{N})$ be the real Hilbert space of squared-summable sequences. Define the right-shift operator S on $\ell^2(\mathbb{N})$ by

$$S(x)_k = \begin{cases} x_{k-1} & \text{if } k \geq 2 \\ 0 & \text{if } k = 1, \end{cases}$$

where $x = (x_k)_{k=1}^{+\infty}$ is in $\ell^2(\mathbb{N})$. Prove the following facts:

(a) $\|S\| = 1$.

(b) The point spectrum of S is empty.

(c) $\sigma(S) = [-1, 1]$.

12. Define the left-shift operator T on $\ell^2(\mathbb{N})$ by

$$T(x)_k = x_{k+1} \qquad \text{for all } k \geq 1,$$

where $x = (x_k)_{k=1}^{+\infty}$ is in $\ell^2(\mathbb{N})$. Prove the following facts:

(a) $\|T\| = 1$.

(b) The point spectrum of T is given by $(-1, 1)$.

(c) $\sigma(T) = [-1, 1]$.

13. Let $H = L^2([-\pi, \pi])$ and let $K : H \to H$ be defined by

$$(Tf)(x) = \int_{-\pi}^{\pi} |x - y| f(y) dy.$$

Using the Fourier's series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

find the spectrum of T.

14. Solve the following Fredholm integral equation for $u(x)$ in $L^2([0, 2\pi])$:

$$u(x) = \cos x + \lambda \int_0^{\pi} \sin(x - y) u(y).$$

15. Solve the following Fredholm integral equation for $u(x)$ in $L^2([0, 1])$:

$$u(x) = e^x + \lambda \int_0^1 (5x^2 - 3) y^2 u(y).$$

16. Discuss the existence of solutions to the following integral equation

$$u(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+y)u(y)dy$$

for the two cases

(a) $\lambda = 1/\sqrt{\pi}$, $f(x) = x^2$,

(b) $\lambda = 1/\pi$, $f(x) = \sin(3x)$

6.7 *Envisaged outcomes*

At the end of this chapter the student should

- Be familiar with the notion of adjoint operator and the definition of self-adjointness for bounded operators. Know the definition of unitary operator.
- Know the basics of Fredholm's alternative principle for closed range operators.
- Know definition and examples of nonnegative operators, know how to characterize the norm of a self-adjoint operator via its quadratic form.
- Be familiar with the uniform boundedness principle on Hilbert spaces and its applications to weak convergence.
- Be familiar with Banach-Alaoglu's theorem on the weak compactness of the unit ball in a Hilbert space.
- Be familiar with the definition of resolvent, spectrum, eigenvalue, continuum spectrum, and residual spectrum of a bounded linear operator on a Hilbert space.
- Know the main properties of the resolvent and the spectrum.
- Know the definition of spectral radius and be able to characterize it using the operator norm.
- Be familiar with the structure of the spectrum of a compact self-adjoint operator on a Hilbert space.
- In the exercises, the student should be able to recognize compact operators on a Hilbert space and to describe their spectrum in some special cases.
- The student should be able to describe the spectrum of special class of operators such as shift operators on sequence spaces, integral operators of Volterra type, multiplication operators, simple convolution operators.
- Be familiar with the main properties of Hilbert-Schmidt operators.
- In the exercises, be able to solve some classes of integral equations using Fredholm's alternative principle.

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